問題:

令 F_n 爲費伯那西數列 $(F_1=1, F_2=1, F_{n+1}=F_n+F_{n-1}, n\geq 2)$, 試求

$$\sum_{n=2}^{\infty}\frac{1}{F_{n-1}F_{n+1}}.$$

解答:

設 k 爲自然數, 且 $k\geq 2$, 則

$$\begin{aligned}
\sum_{n=2}^{k}\frac{1}{F_{n-1}F_{n+1}} &= \sum_{n=2}^{k}\frac{1}{F_{n+1}-F_{n-1}}\left(\frac{1}{F_{n-1}}-\frac{1}{F_{n+1}}\right)\\
&= \sum_{n=2}^{k}\frac{1}{F_n}\left(\frac{1}{F_{n-1}}-\frac{1}{F_{n+1}}\right)\\
&= \sum_{n=2}^{k}\left(\frac{1}{F_{n-1}F_n}-\frac{1}{F_nF_{n+1}}\right)\\
&= \left(\frac{1}{F_1F_2}-\frac{1}{F_2F_3}\right)+\left(\frac{1}{F_2F_3}-\frac{1}{F_3F_4}\right)+\cdots+\left(\frac{1}{F_{k-1}F_k}-\frac{1}{F_kF_{k+1}}\right)\\
&= \frac{1}{F_1F_2}-\frac{1}{F_kF_{k+1}}.
\end{aligned}$$

顯然 $F_n\geq n, \forall n\in\mathbb{N}$, 因此 $0<\frac{1}{F_kF_{k+1}}\leq\frac{1}{k(k+1)}$, 且由 $\lim\limits_{k\to\infty}\frac{1}{k(k+1)}=0$ 與夾擠定理, 可得 $\lim\limits_{k\to\infty}\frac{1}{F_kF_{k+1}}=0$.
故

$$\begin{aligned}
\sum_{n=2}^{\infty}\frac{1}{F_{n-1}F_{n+1}} &= \lim_{k\to\infty}\left(\sum_{n=2}^{k}\frac{1}{F_{n-1}F_{n+1}}\right)\\
&= \lim_{k\to\infty}\left(\frac{1}{F_1F_2}-\frac{1}{F_kF_{k+1}}\right)\\
&= \frac{1}{F_1F_2}-0\\
&= 1.
\end{aligned}$$