Entrance exam for admission in MPSI preparatory class

2011 Session

Allowed time：4 hours

The following eight problems are mutually independent, covering various topics and can be treated in any order. Difficulty is more or less increasing. Calculators are not allowed. Your answers shall be written either in French or English.

1. Find x in R such that $\cos x = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$.

2. Let x be a nonzero real number, such that $x+\frac{1}{x}\in Z$. Prove that $x^n+\frac{1}{x^n}\in Z$ for every $n\in N$.

3. (a) Prove that for every $n\in N^*$, $1\cdot 1!+2\cdot 2!+\cdots+n\cdot n! = (n+1)!-1$.
   (b) Let A be an integer such that $0\le A\le (n+1)!-1$. Prove that one can find integers $a_1,\cdots,a_n$ satisfying $0\le a_k\le k$ for every k, and such that $A = a_1\cdot 1!+\cdots+a_n\cdot n!$.
   (c) Prove that this decomposition is unique.

4. Let $n\in N^*$ and $x_1,\cdots,x_n$ be nonzero real numbers, such that $(n-1)\sum_{k=1}^{n} {x_k}^2 \le (\sum_{k=1}^{n} x_k)^2$. Prove that $x_1,\cdots,x_n$ all have the same sign.

5. Let ABC be a triangle, $a = BC$, $b = CA$ and $c = AB$. Denote by α,β and γ the angles at each respective vertex A,B and C. Finally, let S be the surface area of ABC.
   (a) Find an expression for a^2, in terms of $(b-c)^2$,S and $\tan(\alpha/2)$.
   (b) Prove that the following inequality holds：

$$a^2+b^2+c^2 \ge 4\sqrt{3}S+(a-b)^2+(b-c)^2+(c-a)^2$$

6. Let $\overrightarrow{x_1},\overrightarrow{x_2},\overrightarrow{x_3}$ be unitary vectors in the plane, and assume they are not all three in the same half-plane. Prove that $\left\|\overrightarrow{x_1}+\overrightarrow{x_2}+\overrightarrow{x_3}\right\|\le 1$.

7. Let $ABCD$ be a parallelogram in the plane P. Prove that we can find A',B',C',D' in 3-dimentional space such that $A'B'C'D'$ is a square (and therefore A',B',C',D' sit in a single plane), and such that A(respectively B,C,D) is the orthogonal projection of A' (respectively B',C',D') on P.

8. Let $n\in N$. Prove that

$$\sum_{k=0}^{n}\binom{n+k}{k}2^{-k} = 2^n$$