MPSI Class Entrance Test (2011)

September 2010

Test time：4 hours

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The following exercises can be solved independently and done in any order. The question are listed from easiest to most difficult. Calculators are not permitted. Solutions should be written in French or in English.

1. For $n \in N^*$ let $S_n = \dfrac{1^n + 2^n + \cdots + n^n}{n^n}$. Prove that $S_n \le \dfrac{e}{e-1}$ for all n.
2. Let ABC be a triangle. For which point(s) M in the interior of ABC is the sum of distances from M to the three sides of triangle maximal？
3. Does it exist $n \in N^*$ and n real numbers $a_1, a_2, \cdots, a_n$ such that ,for all real number x,
$$a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx > 0 \text{ ?}$$
4. Is the set of points (x, y) of R^2 such that $\sqrt{x} + \sqrt{y} = 1$ some piece of parabola？Give the graphic shape of the set.
5. Let $f : R \to R$ and $a > 0$ such that, for all real number x,
$f(x+a) = \dfrac{1}{2} + \sqrt{f(x) - f(x)^2}$. Prove that f is a periodic function.
6. If A is a non-empty subset of R, $\hat{A}$ is defined as the set
$$\hat{A} = \{a + a', (a, a') \in A^2\}$$
(a) Let A be a finite subset of cardinal n. Prove that $2n - 1 \le Card\ \hat{A} \le \dfrac{n(n+1)}{2}$

(b) Let n in N^*

Find a subset A of R such that $Card\ A = n$ and $Card\ \hat{A} = 2n - 1$,

a subset B of R such that $Card\ B = n$ and $Card\ \hat{B} = \dfrac{n(n+1)}{2}$.
7. Let a in N^* and n in N^*. Prove that exists b in N^* such that
$$\left(\sqrt{a} - \sqrt{a-1}\right)^n = \sqrt{b} - \sqrt{b-1}\,.$$