2024 年台灣數學奧林匹亞考試試題

比賽日期：2024 年 1 月 31 日

時間限制：四小時 (9:30–13:30)
除作圖外, 答案限用黑色或藍色筆書寫。答案不得以修正液 (帶) 修正。
計算紙必須連同試卷交回。不得使用計算器。
本試卷共五題, 每題滿分七分

問題一. 令 n 與 k 爲正整數。寶寶用 n^2 個數字積木拼成一個 $n \times n$ 的方陣，每塊積木都是一個不超過 k 的正整數。路過的奶爸一看，發現：

1. 方陣上每一橫列的數字都可以視爲以最左方數字爲首項的等差數列，且其公差都不同；
2. 方陣上每一直排的數字都可以視爲以最上方數字爲首項的等差數列，且其公差都不同，

試求 k 的最小可能值 (以 n 的函數表示。)

註： 公差可能非正。

Problem 1. Let n and k be positive integers. A baby uses n^2 blocks to form a $n \times n$ grid, with each of the blocks having a positive integer no greater than k on it. The father passes by and notice that:

1. each row on the grid can be viewed as an arithmetic sequence with the left most number being its leading term, with all of them having distinct common differences;
2. each column on the grid can be viewed as an arithmetic sequence with the top most number being its leading term, with all of them having distinct common differences,

Find the smallest possible value of k (as a function of n.)

Note: The common differences might not be positive.

問題二. 我們稱一個正整數爲傑出數，若其等於 $1, 2, \ldots, n$ 的最小公倍數，其中 n 爲正整數。找出所有符合 $x + y = z$ 的傑出數 x、y、z。

Problem 2. A positive integer is *superb* if it is the least common multiple of $1, 2, \ldots, n$ for some positive integer n. Find all superb x, y, z such that $x + y = z$.

問題三. 試求所有從實數映至實數的函數 f，滿足：

$$2f\left((x+y)^2\right) = f(x+y) + (f(x))^2 + (4y-1)f(x) - 2y + 4y^2$$

對於所有實數 x 和 y 皆成立。

Problem 3. Find all functions f from real numbers to real numbers such that

$$2f\left((x+y)^2\right) = f(x+y) + (f(x))^2 + (4y-1)f(x) - 2y + 4y^2$$

holds for all real numbers x and y.

問題四. 設 O 爲三角形 ABC 的外心。令 E, $F \neq A$ 分別爲線段 CA, AB 上的點，P 爲一點滿足 $\overline{PB} = \overline{PF}$ 且 $\overline{PC} = \overline{PE}$。設直線 OP 分別交 CA, AB 於 Q, R，過 P 且垂直於 EF 的直線分別交 CA, AB 於 S, T。證明：Q, R, S, T 四點共圓。

Problem 4. Suppose O is the circumcenter of $\triangle ABC$ and E, F are points on segments CA and AB respectively with E, $F \neq A$. Let P be a point such that $PB = PF$ and $PC = PE$. Let OP intersect CA and AB at points Q and R respectively. Let the line passing through P and perpendicular to EF intersect CA and AB at points S and T respectively. Prove that points Q, R, S, and T are concyclic.

問題五. 對於平面上有限多個三角形所成的集合，若其中任兩個三角形內部的交集非空，則稱這個三角形集合**相交**。證明：對於平面上兩組相交的三角形集合，必存在一條直線同時通過兩個集合中所有三角形的內部。

Problem 5. A finite collection of triangles on the plane are *intersecting* if, for any two triangles in the collection, the intersection of their interiors is not empty. Show that for any two finite collections of intersecting triangles, there exists a straight line that simultaneously intersects the interiors of all triangles in both collections.