2024 年亞太數學奧林匹亞競賽 初選考試 (一) 試題

考試時間：2023 年 11 月 11 日上午 10:00 ∼ 12:00

說明：本試題共兩頁，分成兩部分：選填題與非選擇題。

作答方式：

- 選填題用 2B 鉛筆在「答案卡」上作答；更正時應以橡皮擦擦拭，切勿使用修正液（帶）。
- 非選擇題用藍、黑色原子筆在「答案卷」上作答；更正時可使用修正液（帶）。
- 未依規定畫記答案卡，致機器掃描無法辨識答案，或未使用藍、黑色原子筆書寫答案卷，致評閱人員無法辨認答案者，其後果由考生自行承擔。
- 不得使用量角器、計算器及其他電子設備。
- 答案卷每人一張，不得要求增補。

第一部分：選填題

說明：本部分共有五題，每題七分，答錯不倒扣，未完全答對不給分。

答案卡填答注意事項：答案的數字位數少於填答空格數時，請適當地在前面填入 0。

1. 老師和另外 20 位同學圍成一圈。老師戴著一頂帽子，並且要分配帽子給另外五位同學，使得任兩個有帽子的人（含老師）之間都至少有兩個沒有帽子的人。則老師一共有 ①② 種不同的分配帽子方式。
2. 2024^{2024} 除以 102 的餘數是 ③④⑤。
3. 考慮多項式函數 $P(x)=x^2-1$。設實數 a 滿足

$$P(P(P(a)))=2024.$$

將 a^2 之值寫成 $m+\sqrt{n}$，其中 m,n 爲正整數且 n 不被大於 1 的平方數整除，則 $m+n$ 之值爲 ⑥⑦。

4. 在 $\triangle ABC$ 中，已知 $\angle A=62^\circ$，$\angle B=64^\circ$，且 D,E 分別爲通過 C,B 的高的垂足。直線 DE 與 $\triangle ABC$ 外接圓有兩個交點，在 AB 劣弧上的稱爲 P，在 AC 劣弧上的稱爲 Q。則 $\angle QAE-\angle PAD=$ ⑧⑨ $^\circ$。

5. 已知 x 是一個三位數的正整數且 x^2 除以 2024 的餘數爲 1，則 x 的最小可能值爲 ⑩⑪⑫。

第二部分：非選擇題

說明：每題 7 分。答案必須寫在「答案卷」上，並標明題號，同時必須寫出演算過程或理由，否則將予扣分甚至零分。作答使用藍、黑色原子筆書寫，除幾何作圖外不得使用鉛筆。若因字跡潦草、未標示題號、標錯題號等原因，致評閱人員無法清楚辨識，其後果由考生自行承擔。

一、 有 2024 位妹妹各帶來一隻洋娃娃。考慮將這 2024 隻洋娃娃分給這些妹妹，使得每位妹妹各揹一隻洋娃娃的所有配對方法。對於所有 $k \geq 0$，令 $p(k)$ 爲這些方法中，恰有 k 位妹妹揹著自己帶來的洋娃娃的方法數量。證明：

$$\sum_{k=0}^{2024} k \cdot p(k) = 2024!.$$

二、 設 x, y, z 爲正實數。試求

$$\frac{x}{3x+y+z} + \frac{y}{x+3y+z} + \frac{z}{x+y+3z}$$

之值的範圍。

2024 APMO Taiwan Preliminary Round 1

10:00–12:00, November 11, 2023

General instructions.

- There are 2 pages of problems, consisting of fill-in problems and non-multiple-choice problems.
- Use 2B pencils to answer fill-in problems on the designated card. Use erasers only to make corrections for these, do not use correction tape/fluid.
- Use pens in blue or black ink to answer non-multiple-choice problems on the designated sheet of paper. Correction tape/fluid may be used to make corrections for this part.
- Contestants are held responsible for the consequences from failing to follow the instructions above so that the machine cannot read the designated card, or the answers for non-multiple-choice problems are illegible.
- Protractors, calculators and other electronic devices are prohibited.
- One sheet of paper for the non-multiple-choice problems is given to each contestants. No more supply is offered.

Part 1. Fill-in problems

Instruction There are FIVE problems in this Part. Each problem is worth 7 points. There is no penalty for wrong answers. No marks will be awarded for answers that are not completely correct.

If the number of digits for the answers is less than the number of designated spaces, fill in a proper number of 0's at the beginning of your answer.

1. A teacher and 20 students stand in a circle. The teacher has a hat. She also needs to distribute hats to five different students so that there are at least two people without hat between each pair of people with hats (including the teacher.) Then the teacher has ①② different ways to distribute the hats.
2. 2024^{2024} divided by 102 leaves a remainder of ③④⑤ .

3. Consider the polynomial $P(x) = x^2 - 1$. There is a real number a that satisfies

$$P(P(P(a))) = 2024.$$

Write a^2 as $m + \sqrt{n}$, where m and n are integers and n is square-free. The value of $m+n$ is $\underline{⑥⑦}$.

4. Let $\triangle ABC$ be a triangle with $\angle A = 62^\circ$ and $\angle B = 64^\circ$. Points D and E are the feet of altitudes that pass through C and B, respectively. Line DE intersects the circumcircle of $\triangle ABC$ at two points: the point that lies on the inferior arc AB is called P, and the point that lies on the inferior arc AC is called Q. Then $\angle QAE - \angle PAD = \underline{⑧⑨}^\circ$.

5. If x is a three-digit integer and x^2 divided by 2024 leaves a remainder of 1, then the smallest possible value of x is $\underline{⑩⑪⑫}$.

Part 2. Non-multiple-choice problems

$\boxed{\textbf{Instruction}}$ There are TWO problems in this part. Each problem is worth 7 points. Answers should be written in blue or black ink, except for graphics that can be drawn by pencil. The problem number should be indicated clearly. The intermediate steps and reasons should be clearly stated, or penalty in deduction of points will be incurred.

I. There are 2024 girls each of who brings her own doll. Consider all the ways to distribute these dolls, one to each of the 2024 girls. For any $k \geq 0$, let $p(k)$ be the number of ways to distribute the dolls so that there are exactly k girls received her own doll. Prove that

$$\sum_{k=0}^{2024} k \cdot p(k) = 2024!.$$

II. Let x, y, z be three positive real numbers. Determine all possible values for the expression

$$\frac{x}{3x+y+z} + \frac{y}{x+3y+z} + \frac{z}{x+y+3z}.$$

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選擇題參考解答

1. 答案：56

   解答：令每組相鄰帽子之間的人數爲 $x_i,\ i=1\backsim 6$，則我們有 $\sum_{i=1}^{6} x_i = 15$ 且 $x_i \geq 2$，等價於 $\sum_{i=1}^{6}(x_i-2) = 15-12 = 3$。由重複組合知這樣的解共有 $C_5^{6+3-1} = C_5^8 = 56$。

2. 答案：052

   解答：注意到 $102 = 2\times 3\times 17$。且我們有 $2024^{2024} \equiv 0 \mod 2$，$2024^{2024} \equiv (-1)^{2024} \equiv 1 \mod 3$ 和 $2024^{2024} \equiv (1)^{2024} \equiv 1 \mod 17$，故由中國剩餘定理知 2024^{2024} 除以 102 的餘數爲 52。

3. 答案：47

   解答：由 $P(P(P(a))) = 2024$，得

   $$P(P(a))^2 - 1 = 2024 \Rightarrow P(P(a)) = \pm\sqrt{1+2024} = \pm 45.$$

   但負不合，因爲 P 的值域爲 $[-1,\infty)$。再操作一次可得

   $$P(a)^2 - 1 = 45 \Rightarrow (a^2-1)^2 = 46 \Rightarrow a^2 = 1+\sqrt{46}.$$

   故 $m+n = 1+46 = 47.$

4. 答案：10°

   解答：可知 $DBCE$ 共圓，所以 $\angle ADE = 54^\circ, \angle AED = 64^\circ$。因此有 $\overset{\frown}{AP} + \overset{\frown}{PB} = \overset{\frown}{AQ} + \overset{\frown}{PB}$ 所以 $\angle APD = \angle AQE$ 容易得知 $\angle QAE - \angle PAD = 10^\circ$

5. 答案：461

   由於 $2024 = 8\times 11\times 23$，因此 $x^2 \equiv 1 \mod 2024$ 若且唯若 $x \equiv \pm 1 \mod 11$，$x \equiv \pm 1 \mod 23$ 且 x 爲奇數。這表示 $x \equiv \pm 1$ or $\pm 45 \mod 2\times 11\times 23$。逐一檢查後得最小的三位數字爲 $x = 506-45 = 461$。

2024 年亞太數學奧林匹亞競賽 初選考試 (一)

非選擇題參考解答

一、 原命題等價於對 $\{1, 2, \cdots, 2024\}$ 的重排，$p(k)$ 等價於恰有 k 個不動點的重排數量。注意到等號左式爲所有重排的不動點總數。又注意到讓 1 爲不動點的重排數爲 $(2024-1)!$，其餘點雷同，因此所有重排的不動點總數必須爲 $2024 \times (2024-1)! = 2024!$，故等式成立。

二、 首先注意到要求的式子是 x, y, z 的齊次式，故不失一般性可設 $x+y+z=2$，使原式成爲

$$\begin{aligned}
&\frac{x}{3x+y+z}+\frac{y}{x+3y+z}+\frac{z}{x+y+3z}\\
=&\frac{x}{2x+2}+\frac{y}{2y+2}+\frac{z}{2z+2}\\
=&\frac{1}{2}\left(\frac{x}{x+1}+\frac{y}{y+1}+\frac{z}{z+1}\right)\\
=&\frac{3}{2}-\frac{1}{2}\left(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right).
\end{aligned}$$

由柯西不等式知

$$\begin{aligned}
\left(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right)((x+1)+(y+1)+(z+1)) &\geqslant (1+1+1)^2,\\
\left(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right)\cdot 5 &\geqslant 9,\\
\left(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right) &\geqslant \frac{9}{5},
\end{aligned}$$

所以

$$\frac{x}{3x+y+z}+\frac{y}{x+3y+z}+\frac{z}{x+y+3z} \leqslant \frac{3}{2}-\frac{1}{2}\cdot\frac{9}{5}=\frac{3}{5},$$

且等號發生於 $x=y=z=\dfrac{2}{3}$ 時。

另一方面，利用調整法可知當 x,y,z 之間的距離越大時，所求之值會越小。極端情形爲 $(x,y,z)=(2,0,0)$ (或其排列)，其值爲 $1/3$。因爲這個最小值永遠達不到但又可以任意靠近，故所求值域爲區間 $\left(\dfrac{1}{3}, \dfrac{3}{5}\right]$。