臺北市立西松高級中學 111 學年度教師甄選數學科試題
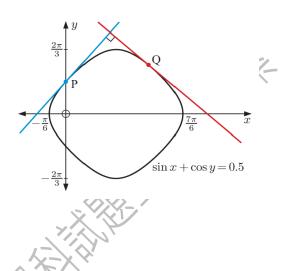
Question 1. [Maximum mark: 6]

The relation $\sin x + \cos y = 0.5$ is graphed alongside for

$-\dfrac{\pi}{6} \le x \le \dfrac{7\pi}{6}$, $-\dfrac{2\pi}{3} \le y \le \dfrac{2\pi}{3}$. P lies on the y-axis, and the

tangents at P and Q are perpendicular. The coordinates of Q

is _____ .

Ans: $Q(\dfrac{5\pi}{6}, \dfrac{\pi}{2})$

Question 2. [Maximum mark: 6]

Jack and John have decided to play a game. One roll of a die is considered as one round of the game. On each round, John agrees to pay Jack \$4 if 1 or 2 is rolled, Jack agrees to pay John \$2 if 3, 4, 5 or 6 is rolled, and who is paid wins the round. In the end, who earns money wins the game. Now Jack and John roll a die 12 times. Let A denote the number of rounds Jack wins. Find the expected value of A, E[A]= _____ .

Ans: 4

Question 3. [Maximum mark: 9]

A function $f(x)$ is defined by $f(x) = arc\cos\left(\dfrac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$ and a function $g(x)$ is defined by

$g(x) = arc\cos\left(\dfrac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$, $x \ge 0$.

(a) Find the equation of the horizontal asymptote to the graph of $y = f(x)$. [3]

(b) Find $f'(x)$ for $x \in \mathbb{R}$, $x \ne 0$. [3]

(c) Find the expression for $g^{-1}(x)$, and state the domain of $g^{-1}(x)$. [3]

Ans:

(a) $y = 0$ is a horizontal asymptote for the graph of $y = f(x)$.

(b) $f'(x) = -\dfrac{2x}{\sqrt{x^2(x^2+1)}}$ for $x \in \mathbb{R}$, $x \ne 0$. $x \in \square$, $x \ne 0$.

(c) $g^{-1}(x) = \sqrt{\dfrac{1+\cos x}{1-\cos x}}$ and its domain is $(o, \pi]$.

Question 4. [Maximum mark: 10]

An estate manager is responsible for stocking a small lake with fish. He begins by introducing 1000 fish into the lake and monitors their population growth to determine the likely carrying capacity of the lake. After one year an accurate assessment of the number of fish in the lake is taken and it is found to be 1200. Let N be the number of fish t years after the fish have been introduced to the lake.

Initially it is assumed that the rate of increase of N will be constant.

When $t=8$ the estate manager again decides to estimate the number of fish in the lake. To do this he first catches 300 fish and marks them, so they can be recognized if caught again. These fish are then released back into the lake. A few days later he catches another 300 fish, releasing each fish after it has been checked, and finds 45 of them are marked.

Assuming the proportion of marked fish in the second sample is equal to the proportion of marked fish in the lake. Let X be the number of marked fish caught in the second sample, where X is considered to be distributed as $B(n,p)$.

(a) Write down the value of n and the value of p. [2]

(b) State an assumption that is being made for X to be considered as following a binomial distribution. [2]

(c) Show that the variance of the proportion of marked fish in the sample, Var($X/300$), is 0.000425. [2]
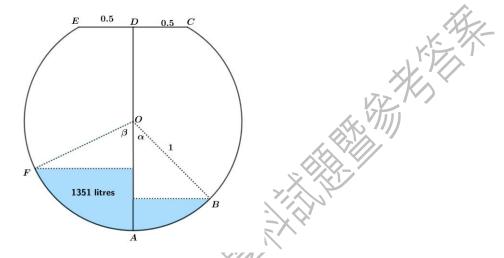
The estate manager feels confident that the proportion of marked fish in the lake will be within 1.5 standard deviations of the proportion of marked fish in the sample and decides these will form the upper and lower bounds of his estimate.

(d) Taking the value for the variance given in (c) as a good approximation for the true variance, find upper and lower bounds for the number of fish in the lake when $t=8$. [2]

(e) Given this result, comment on the validity of the linear model used initially. [2]

Question 5. [Maximum mark: 9]

A water truck tank which is 3 meters long has a uniform cross-section in the shape of a major segment. The tank is divided into two equal parts and is partially filled with water as show in the following diagram of the cross-section. The centre of the circle is O, the angle AOB is $\boldsymbol{\alpha}$ radians, and the angle AOF is $\boldsymbol{\beta}$ radians.

(a) Find an expression for the volume of water V, in m^3, in the left part of the water tank in terms of $\boldsymbol{\beta}$.

[3]

The left part of the tank is now being filled with water at a constant rate of 0.001 m^3 per second.

(b) Calculate $\dfrac{\mathrm{d}\beta}{\mathrm{d}t}$ when $\beta = \dfrac{3\pi}{5}$ (note: $\sin\dfrac{\pi}{5} \approx 0.5878$).

[3]

(c) Calculate the amount of time it will take for the left part of the tank to be fully filled with water. Give your answer in minutes and correct to the nearest integer.

[3]

6.高斯符號$[a]$代表不大於a的最大整數，$[12.7]=12$；$[-2.8]=[-3+0.2]=-3$。

已知方程式$[x]+[2x]+[4x]=102$解集合為A。若$a\in A$，求$[4a+1]$之值? (6分)

Ans：60

7. *Fibonacci*數列$\{a_n\}$的定義如下：$\begin{cases} a_1=a_2=1 \\ a_{n+2}=a_{n+1}+a_n \end{cases}$，其中$n\in N$。

若$m=5k$，其中$k\in N$，試證：$a_m=a_{5k}$必為5之倍數。 (6分)

8.已知$\triangle ABC$中$\overline{BC}=a$，$\overline{CA}=b$，$\overline{AB}=c$。若$\tan\dfrac{C}{2}(b\tan A+a\tan B)=(b+a)$。

試判斷$\triangle ABC$的形狀為何種三角形? (6分)

Ans：等腰三角形

9.銳角$\triangle ABC$中，過A作垂直線交$\overline{BC}$於D，在$\overline{AD}$上取一點P，連接$\overline{CP}$，$\overline{BP}$

分別交$\overline{AB}$，$\overline{AC}$於E，F。試證：$\overline{AD}$為$\angle EDF$之角平分線。 (6分)

10.若$\dfrac{4}{5}\le x\le 3$，求$\sqrt{3-x}+\sqrt{5x-4}$之最大值與最小值? 此時對應之x值? (8分)

Ans：$x=\dfrac{79}{30}\Rightarrow Max=\sqrt{\dfrac{66}{5}}$ ； $x=\dfrac{4}{5}\Rightarrow min=\sqrt{\dfrac{11}{5}}$

11.有一個底面為三角形的直立稜柱，某一彼此相鄰的三個面(即兩個側面和一個底面)的面積總和為一給定值P。試求此直立三角柱體積之最大值?並求出此時柱體的高與底面三角形的形狀? (8分)

Ans：體積之最大值$\dfrac{P\sqrt{P}}{8}$，其中底為腰長$\sqrt{P}$之等腰$\triangle$，柱高$\dfrac{\sqrt{P}}{4}$

12.已知袋中有1~30號的球，一次取四球，問滿足下列條件的情形有幾種?
 (1)取出數字由小到大排列，其數字差距均大於7。例如(2,10,19,28) (3分)
 (2)取出四顆中數字是2的倍數恰有兩顆，是5的倍數也恰有兩顆。
   例如(6,10,19,25) (3分)
 (3)取出四顆中數字是2的倍數恰有一顆，是3的倍數也恰有一顆，且是5的倍數也恰有一顆。例如(6,13,19,25) (4分)
Ans：(1)126   (2)1692   (3)1240

13.IB 常識題，請以英文作答。（10%）

International mindedness lies at the very heart of the IB mission, which aims to develop students 'who help create a better and more peaceful world.' International mindedness is an overarching idea. Please indicate three dimensions of being internationally-minded in DP programs. In addition, please elaborate on ways to facilitate international mindedness in your subject.