性質 1

$[x]$高斯函數

$n,k\in N,n\geq k$則$\left[\frac{n+1}{k}\right]-\left[\frac{n}{k}\right]=\begin{cases}1 & n+1=tk\\ 0 & n+1=tk+r\end{cases},t\in N\cup\{0\},r\in N,r<k$

Proof：

$n,k\in N,n\geq k\Rightarrow\exists t,r\in N\cup\{0\},r<k\ such\ that\ n=tk+r,$

$\Rightarrow\left[\frac{n}{k}\right]=\left[t+\frac{r}{k}\right]=t,$

$\left[\frac{n+1}{k}\right]=\left[t+\frac{r+1}{k}\right]=\begin{cases}t+1 & r+1=k\\ t & r+1<k\end{cases},n+1=tk+(r+1)$

$\Rightarrow\left[\frac{n+1}{k}\right]-\left[\frac{n}{k}\right]=\begin{cases}1 & n+1=(t+1)k\\ 0 & tk<n+1<(t+1)k\end{cases}$

性質 2

$n\in N,F(n)=n$的正因數個數$,F(n)=\begin{cases}\text{奇數} & n\text{為完全平方數}\\ \text{偶數} & n\text{非完全平方數}\end{cases}$

Proof

$n=1,\text{trivial}$

$n>1,\exists a_1,\cdots,a_n\in N,p_1,\cdots,p_n$均為質數，使得$n=\prod_{k=1}^{n}p_k^{a_k}$

$F(n)=\prod_{k=1}^{n}(a_k+1)\Rightarrow F(n)$為奇數若且唯若$\forall k\in N,1\leq k\leq n,a_k+1$均為奇數

$\Rightarrow a_k$均為偶數

性質 3

$[x]$高斯函數

$$n\in N,\left[\sqrt{n+1}\right]-\left[\sqrt{n}\right]=\begin{cases}1 & n+1\text{為完全平方數}\\ 0 & n+1\text{非完全平方數}\end{cases}$$

Proof

$$if\ \ n+1=k^2,k\in N\ \ then\ \left[\sqrt{n+1}\right]=\left[\sqrt{k^2}\right]=k$$

$$\left[\sqrt{n}\right]=\left[\sqrt{k^2-1}\right]\Rightarrow\sqrt{k^2-1}-1<\left[\sqrt{n}\right]\le\sqrt{k^2-1}\Rightarrow\sqrt{(k-1)^2}-1<\left[\sqrt{n}\right]<k$$

$$\Rightarrow k-2<\left[\sqrt{n}\right]<k\Rightarrow\sqrt{n}=k-1$$

$$\Rightarrow\left[\sqrt{n+1}\right]-\sqrt{n}=1$$

$$if\ \ n+1\ne k^2\Rightarrow let\ n+1=k^2+r\ \ ,r\in N\ \ and\ \ k^2+r<(k+1)^2$$

$$\Rightarrow\left[\sqrt{n+1}\right]=\left[\sqrt{n}\right]=k\Rightarrow\left[\sqrt{n+1}\right]-\left[\sqrt{n}\right]=0$$

證明：

$\forall n\in N,\left[\sqrt{n}\right]+\sum_{k=1}^{n}\left[\frac{n}{k}\right]$均為偶數，$[x]$高斯函數

令$a_n=\left[\sqrt{n}\right]+\sum_{k=1}^{n}\left[\frac{n}{k}\right],$ 。

$n\in N,$定義$F(n)=n$的正因數個數, $G(n+1)=\left[\sqrt{n+1}\right]-\left[\sqrt{n}\right]$

$$\begin{aligned}a_{n+1}-a_n&=\left[\sqrt{n+1}\right]-\left[\sqrt{n}\right]+\sum_{k=1}^{n}\left(\left[\frac{n+1}{k}\right]-\left[\frac{n}{k}\right]\right)+\left[\frac{n+1}{n+1}\right]\\&=G(n+1)+\left(F(n+1)-1\right)\times1+1\ \ (\text{By 性質1})\\&=G(n+1)+F(n+1)=\text{偶數(By 性質2 \& 性質3)}\end{aligned}$$

又$a_1=\left[\sqrt{1}\right]+\sum_{k=1}^{1}\left[\frac{1}{k}\right]=1+1=2$ 為偶數

綜合以上對於$\forall n\in N,\left[\sqrt{n}\right]+\sum_{k=1}^{n}\left[\frac{n}{k}\right]$均為偶數，$[x]$高斯函數