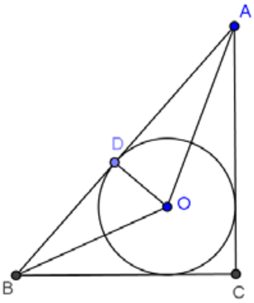
16.直角ΔABC中，$\angle C$是直角，設ΔABC的內切圓 O 交$\overline{AB}$於 D，r 為內切圓 O 的半徑，$\overline{AB}=c$，求證：【100.中正高中二招　★★☆和角公式及倍半角公式】

(1) $r=\dfrac{c}{\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2})}$

(2) $r\le\dfrac{c}{2}(\sqrt{2}-1)$

【解】：



(1)如右圖所示，則$\overline{OA}$、$\overline{OB}$分別為為$\angle A$、$\angle B$的分角線，

圓 O 與$\overline{AB}$切於 D 點，則$\overline{OD}\perp\overline{AB}$，且$\overline{OD}=r$

由 $\cot\frac{A}{2}=\dfrac{\overline{AD}}{r}\Rightarrow\overline{AD}=r\cot\frac{A}{2}$

由 $\cot\frac{B}{2}=\dfrac{\overline{BD}}{r}\Rightarrow\overline{BD}=r\cot\frac{B}{2}=r\cot(45^\circ-\frac{A}{2})$

$\therefore c=\overline{AD}+\overline{BD}=r(\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2}))\Rightarrow r=\dfrac{c}{\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2})}$

(2) $\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2})=\dfrac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\tan(45^\circ+\frac{A}{2})=\dfrac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\dfrac{\tan45^\circ+\tan\frac{A}{2}}{1-\tan45^\circ\tan\frac{A}{2}}$

$=\dfrac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\dfrac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}}=\dfrac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\dfrac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}=\dfrac{\cos\frac{A}{2}(\cos\frac{A}{2}-\sin\frac{A}{2})+\sin\frac{A}{2}(\cos\frac{A}{2}+\sin\frac{A}{2})}{\sin\frac{A}{2}(\cos\frac{A}{2}-\sin\frac{A}{2})}$

$=\dfrac{1}{\sin\frac{A}{2}\cos\frac{A}{2}-\sin^2\frac{A}{2}}=\dfrac{1}{\frac{1}{2}\sin A-\frac{1-\cos A}{2}}=\dfrac{2}{\sin A+\cos A-1}=\dfrac{2}{\sqrt{2}\sin(A+45^\circ)-1}$

$\because 0<A<\frac{\pi}{2}$，$\therefore\frac{\pi}{4}<A+45^\circ<\frac{3\pi}{4}$，故$\frac{\sqrt{2}}{2}<\sin(A+45^\circ)\le1$

$\because\dfrac{1}{\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2})}=\dfrac{\sqrt{2}\sin(A+45^\circ)-1}{2}$，$\therefore\dfrac{1}{\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2})}\le\dfrac{\sqrt{2}-1}{2}$

$\therefore\dfrac{c}{\cot\frac{A}{2}+\cot(45^\circ-\frac{A}{2})}\le\dfrac{c(\sqrt{2}-1)}{2}\Rightarrow r\le\dfrac{c(\sqrt{2}-1)}{2}$