109 板中

一、填充題(每題6分，共66分)

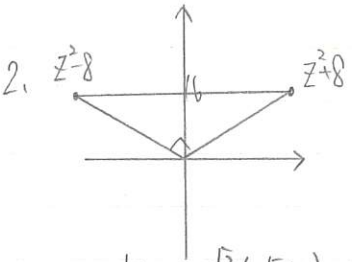
1. 令 $f(x)=ax^3+bx^2+cx+d$, $f(\frac{1}{3})=a\cdot(\frac{1}{3})^3+b\cdot(\frac{1}{3})^2+c\cdot(\frac{1}{3})+d$ $\rightarrow f(\frac{1}{3})+f(-\frac{1}{3})=\frac{2}{9}b+2d=120d$

$f(-\frac{1}{3})=a\cdot(-\frac{1}{3})^3+b\cdot(-\frac{1}{3})^2+c\cdot(-\frac{1}{3})+d$

$\hookrightarrow \frac{2}{9}b=118d$

所求 $\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}=\frac{-b/a}{-d/a}=\frac{b}{d}=118\times\frac{9}{2}=59\times 9=531$ #

2.



$\frac{5\pi}{6}=150^\circ$, $\frac{\pi}{3}=60^\circ$

令 $|z^2+8|=r_1$, $|z^2-8|=r_2$ $\quad r_1\cos60^\circ+r_2\cos30^\circ=16 \Rightarrow \frac{1}{2}r_1+\frac{\sqrt{3}}{2}r_2=16$.

$r_1\sin60^\circ=r_2\sin30^\circ \Rightarrow \frac{\sqrt{3}}{2}r_1=\frac{1}{2}r_2$

故 $\frac{1}{2}r_1+\frac{\sqrt{3}}{2}(\sqrt{3}r_1)=16 \Rightarrow \frac{1}{2}r_1+\frac{3}{2}r_1=16$, $2r_1=16$, $r_1=8$, $r_2=8\sqrt{3}$

令 $z^2=(x+yi)$, $z^2+8=(x+8)+yi \rightarrow (x+8)^2+y^2=64$

$z^2-8=(x-8)+yi \rightarrow (x-8)^2+y^2=192$ $\Rightarrow 32x=-128$

$x=-4$, $y=4\sqrt{3}$.

$z^2=(-4+4\sqrt{3}i)$, $|z^2|=\sqrt{16+48}=\sqrt{64}=8=|z|^2$, $|z|=2\sqrt{2}$

$\hookrightarrow \begin{cases}\cos\theta=-\frac{1}{2}=2\cos^2\frac{1}{2}\theta-1 \Rightarrow \cos^2\frac{1}{2}\theta=\frac{1}{4}\\ \sin\theta=\frac{\sqrt{3}}{2}\\ \quad =2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\end{cases}$

$\hookrightarrow \cos\frac{1}{2}\theta=\pm\frac{1}{2}$

$\sin\frac{1}{2}\theta=\frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$

$z=2\sqrt{2}(\frac{1}{2}+\frac{\sqrt{3}}{2}i)$ 或 $2\sqrt{2}(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)$

$=\sqrt{2}+\sqrt{6}i$ 或 $-\sqrt{2}-\sqrt{6}i$ #

3. $(1+i)^n=a_n+ib_n$

$(1+i)^{n+1}=a_{n+1}+ib_{n+1}=(a_n+ib_n)(1+i)=(a_n-b_n)+(a_n+b_n)i$ $\Rightarrow T=\begin{pmatrix}1&-1\\1&1\end{pmatrix}$, $T^2=\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$

$=\begin{pmatrix}0&-2\\2&0\end{pmatrix}$

$T^4=\begin{pmatrix}0&-2\\2&0\end{pmatrix}\begin{pmatrix}0&-2\\2&0\end{pmatrix}=\begin{pmatrix}-4&0\\0&-4\end{pmatrix}$, $T^8=\begin{pmatrix}16&0\\0&16\end{pmatrix}$, $T^{16}=\begin{pmatrix}256&0\\0&256\end{pmatrix}$

$T^{26}=T^2\cdot T^8\cdot T^{16}=\begin{pmatrix}0&-2\\2&0\end{pmatrix}\begin{pmatrix}16&0\\0&16\end{pmatrix}\begin{pmatrix}256&0\\0&256\end{pmatrix}=\begin{pmatrix}0&-2\\2&0\end{pmatrix}\begin{pmatrix}4096&0\\0&4096\end{pmatrix}=\begin{pmatrix}0&-8192\\8192&0\end{pmatrix}$ #

4. $f(x) \overset{let}{=} 0$, $x^4-2x^3+x^2-2x=0$, $x(x^3-2x^2+x-2)=0$, $x(x-2)(x^2+1)=0$, $x=0$ 或 2

$$\int_a^b f(x)\,dx = -\frac15 x^5+\frac12 x^4-\frac13 x^3+x^2 \Big|_{x=0} = 0$$

$$\Big|_{x=2} = -\frac{32}{5}+8-\frac83+4=\frac{44}{15} \quad \#$$

5. $\overline{AF}^2=4^2+2^2-2\cdot4\cdot2\cdot\cos120^\circ = 20-16\times(-\frac12)=28$, $\overline{AF}=2\sqrt7$, $\overline{PQ}$的 Max. $=2\sqrt7+2$

6. 1⋯1 ↗ 100個1
1⋯1
1⋯1
⋮
1
1⋯890
進10

故末10位為 1234567890 #

7. 令 $\tan x=a$ $(a>0, a\in\mathbb{R})$, $f(x)=\frac1a+15a+25a^2$, $f'(x)=50a+(-a^{-2})+15 \overset{let}{=} 0$

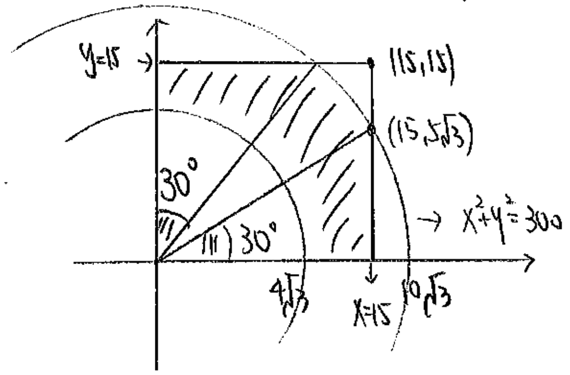
$\Rightarrow 50a^3+15a^2-1=0$, $(5a-1)(\underbrace{10a^2+5a+1}_{D<0})=0$

$a=\frac15$ 時 $f(x)=5+3+1=9$ #

8. ○ ○ ○ ○ ○ (↑ ↑ ↑ ↑ ↑ ↑) $\underbrace{[C^6_3\times3!]}_{\text{不相鄰}}\times\underbrace{[C^5_1C^4_2C^2_2\times\frac{1}{2!}]}_{\text{分組}}\times\underbrace{[1!\times2!\times2!]}_{\text{組內排}}=7200$

9. 海氏 formula → $\frac12\cdot\sqrt{xy+yz+xz}$

10. $\tan(\angle BPO-\tan\angle APO)=\dfrac{\frac{20}{r}-\frac6r}{1+\frac{20}{r}\times\frac6r}\ge\tan30^\circ=\frac{\sqrt3}{3}$ $(r=\sqrt{x^2+y^2})$ $\Rightarrow r^2-14\sqrt3 r+120\le0$, $\underline{4\sqrt3\le r\le10\sqrt3}$

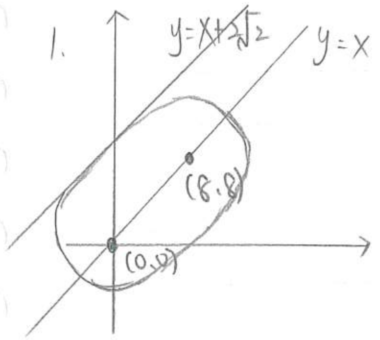


$$\text{area}=\left[15\times5\sqrt3\times\frac12\times2\right]+(10\sqrt3)^2\pi\times\frac{1}{12}-(4\sqrt3)^2\pi\times\frac14$$
$$=75\sqrt3+25\pi-12\pi$$
$$=75\sqrt3+13\pi \quad \#$$

11、 $\frac{1}{2}\times 12^3=\frac{1}{2}\times 144\times 12=144\times 6=864$ #

二、計算題：共22分

1.



$b=\frac{|2\sqrt{2}|}{\sqrt{1+(-1)^2}}=2$ ， $2c=8\sqrt{2}$ ， $c=4\sqrt{2}$ ， $a^2=b^2+c^2=4+32=36$ 、 $a=6$

中心：$(4,4)\rightarrow$ 轉回左右型 $\begin{pmatrix}\cos(-45^\circ) & -\sin(-45^\circ)\\ \sin(-45^\circ) & \cos(-45^\circ)\end{pmatrix}\begin{pmatrix}4\\4\end{pmatrix}$

$=\begin{pmatrix}\frac{\sqrt{2}}{2} & +\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}\begin{pmatrix}4\\4\end{pmatrix}=\begin{pmatrix}4\sqrt{2}\\0\end{pmatrix}$

$\rightarrow$ eq：$\frac{(x-4\sqrt{2})^2}{36}+\frac{(y-0)^2}{4}=1$

$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}\cos 45^\circ & -\sin 45^\circ\\ \sin 45^\circ & \cos 45^\circ\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}y\\ \frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y\end{pmatrix}$

$\Rightarrow\begin{cases}2x'=\sqrt{2}x-\sqrt{2}y\\ 2y'=\sqrt{2}x+\sqrt{2}y\end{cases}\Rightarrow\begin{cases}x=\frac{1}{\sqrt{2}}(x'+y')\\ y=\frac{1}{\sqrt{2}}(y'-x')\end{cases}$

$\Rightarrow$ 所求：$\frac{\left[\frac{1}{\sqrt{2}}(x'+y')-4\sqrt{2}\right]^2}{36}+\frac{\left[\frac{1}{\sqrt{2}}(y'-x')\right]^2}{4}=1$ $(x'\rightarrow x,\ y'\rightarrow y)$

$\Rightarrow\frac{\frac{1}{2}(x^2+y^2+2xy)-8(x+y)+32}{36}+\frac{\frac{1}{2}(x^2+y^2-2xy)}{4}=1$

$\frac{1}{2}(x^2+y^2+2xy)-8(x+y)+32+\frac{9}{2}(x^2+y^2-2xy)=36\Rightarrow 5x^2+5y^2+xy-9xy-8x-8y=4$

$5x^2-8xy+5y^2-8x-8y=4$ #

2、 $x\in\mathbb{Z}\Rightarrow 2x^2+x-2x\cdot(2x)+2(2x^2)=67\Rightarrow 2x^2+x-67=0$ ， $x=\frac{-1\pm\sqrt{537}}{4}\notin\mathbb{Z}$ (✱)

$x>0\Rightarrow$ 令 $x=a+b$ (a為整數部份，b為小數部份) $\rightarrow(a+b)(2(a+b)+1)-2(a+b)\cdot[a+a+1]+2\cdot[a^2+(a+1)^2]=67$

$2a^2+3a+2b^2-b=65$

$a=5$， $50+15+(2b^2-b)=65\rightarrow 2b^2-b=0$， $b(2b-1)=0$， $b=\underset{(不合)}{0}$ 或 $\frac{1}{2}$ $\Rightarrow x=5.5$ #

$x<0\Rightarrow$ 令 $x=a+b$ $\left(\begin{matrix}ex：-7.2\\ a=-8, b=0.8\end{matrix}\right)\rightarrow 2a^2+3a+2b^2-b=65$

$a=-6\rightarrow 72-18=54\ \cdots\times$

$a=-7\rightarrow 98-21=77\ \cdots\times$

$a=-5$， $\underbrace{50-15}_{=35}+2b^2-b\neq 65$

三、證明題：共12分.

1. $a_{n+1}/a_n = \left(1+\frac{1}{n+1}\right)^{n+1} \Big/ \left(1+\frac{1}{n}\right)^{n} = \left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1} \times \left(1+\frac{1}{n}\right)$

$\hookrightarrow \frac{1+\frac{1}{n+1}}{1+\frac{1}{n}} = \frac{\frac{n+2}{n+1}}{\frac{n+1}{n}} = \frac{n(n+2)}{(n+1)^2} = \frac{n^2+2n}{n^2+2n+1} = 1-\frac{1}{n^2+2n+1}$

$\Rightarrow \left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1} \geq 1-(n+1)\times\frac{1}{n^2+2n+1} = 1-\frac{1}{n+1} = \frac{n}{n+1} \Rightarrow \frac{a_{n+1}}{a_n} \geq \frac{n}{n+1}\times\left(1+\frac{1}{n}\right) = 1$

$a_n > 0$，故 $\langle a_n \rangle$ is increasing. #1

let $b_n = \left(1+\frac{1}{n}\right)^{n+1}$, by the similar argument $\Rightarrow \frac{b_{n+1}}{b_n} \leq 1$，故 $\langle b_n \rangle$ is decreasing.

, and we can reply that $a_n \leq b_n \leq b_1$ Hence, $\langle a_n \rangle$ is bounded above by $b_1 = 4$ #2

Bernoulli's inequality: if $x > -1$, then $\begin{cases} (1+x)^r \geq 1+rx & (r \in \mathbb{R}) \text{ for } r \leq 0 \text{ or } r \geq 1 \\ (1+x)^r \leq 1+rx & \text{for } 0 \leq r \leq 1 \end{cases}$