109 興大附中

一、填充題（1~8題4分，9~20題5分）

1. Consider $(1+i)^{2020} = C^{2020}_0 + C^{2020}_1 i + (-1)\cdot C^{2020}_2 + (-i)\cdot C^{2020}_3 + C^{2020}_4 + \cdots + (-i)\cdot C^{2020}_{2019} + C^{2020}_{2020}$

$+\ (1-i)^{2020} = C^{2020}_0 - C^{2020}_1 i + (-1)\cdot C^{2020}_2 + i\cdot C^{2020}_3 + C^{2020}_4 + \cdots + i\, C^{2020}_{2019} + C^{2020}_{2020}$

$$(1+i)^{2020} + (1-i)^{2020} = 2\left(C^{2020}_0 - C^{2020}_2 + C^{2020}_4 - C^{2020}_6 + \cdots - C^{2020}_{2018} + C^{2020}_{2020}\right)$$

$(1+i)^2 = 2i \Rightarrow (1+i)^{2020} = (2i)^{1010} = -2^{1010}$

$(1-i)^2 = -2i \Rightarrow (1-i)^{2020} = (-2i)^{1010} = -2^{1010}$

$\Rightarrow -2\times 2^{1010} = 2$(原式) $\Rightarrow$ 原式 $= -2^{1010}$ #

2. $\begin{cases} a_n + a_{n+1} = b_n \\ a_n \cdot a_{n+1} = \left(\frac{1}{2}\right)^n \end{cases}$

$a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \left(\frac{1}{2}\right),\ a_4 = \left(\frac{1}{2}\right)^2,\ a_5 = \left(\frac{1}{2}\right)^2$

$\lim\limits_{n\to\infty}(b_1 + b_2 + \cdots + b_n) = \lim\limits_{n\to\infty}\left((a_1+a_2) + (a_2+a_3) + \cdots + (a_n + a_{n+1})\right)$

$= \lim\limits_{n\to\infty}\left(1 + 2(a_2 + a_3 + \cdots + a_n) + a_{n+1}\right)$

$= \lim\limits_{n\to\infty}\left(1 + 4\left(\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n\right) + a_{n+1}\right)$

$= 1 + 4\times\dfrac{\frac{1}{2}}{1-\frac{1}{2}} + 0 = 1 + 4 = 5$ #

3. $\sqrt{1+\frac{1}{1}+\frac{1}{4}} = \sqrt{\frac{9}{4}} = \frac{3}{2},\ \sqrt{1+\frac{1}{4}+\frac{1}{9}} = \sqrt{\frac{49}{36}} = \frac{7}{6},\ \cdots,\ \sqrt{1+\frac{1}{2019^2}+\frac{1}{2020^2}} = 1 + \frac{1}{2019\times 2020}$

和 $= 2019 + \sum\limits_{k=1}^{2019} \frac{1}{k(k+1)} = 2019 + \left(1 - \frac{1}{2020}\right) = 2019 + \frac{2019}{2020} = 2019\frac{2019}{2020}$ #

4. $k \in \mathbb{Z} \Rightarrow k^2 - 2k - 19 \neq 0 \Rightarrow y = \dfrac{-(k+2)}{k^2-2k-19}x + \dfrac{(k-9)}{k^2-2k-19}$ Consider $m=0 \Rightarrow k=-2 \Rightarrow y=1$

(OK)

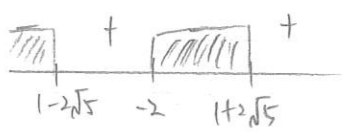
$m>0$ and y截距 >0

$\left(k^2-2k-19 \overset{let}{=} 0,\ k = \frac{2\pm\sqrt{80}}{2} = 1\pm\sqrt{20} = 1\pm 2\sqrt{5}\right)$

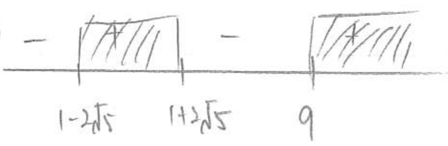
$1+2\sqrt{5} \approx 1+4.4 = 5.4$

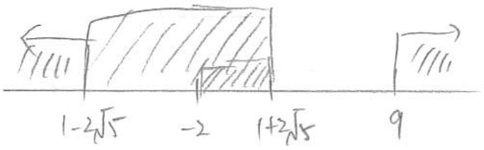
$1-2\sqrt{5} \approx 1-4.4 = -3.4$

$m>0 \Rightarrow (k+2)(k^2-2k-19) < 0$

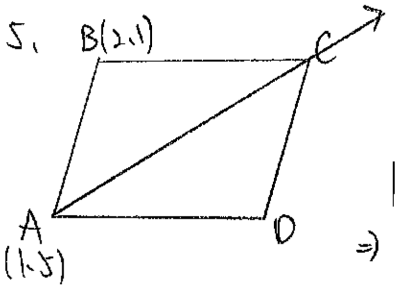


y截距 $>0 \Rightarrow (k-9)(k^2-2k-19) > 0$





故取 $-2 \le x < 1+2\sqrt{5}$ 間整數 $-2,-1,0,1,2,3,4,5 \rightarrow 8$ 個 #

5. 

$3x-y+2=0 \quad \vec{v}=(1,3)$

C可設為 $(1+t, 5+3t)$ for $t \in \mathbb{R} \Rightarrow \overrightarrow{BC}=(t-1, 3t+4)=\overrightarrow{AD}$

$|\overrightarrow{AB}|^2 = 1+16=17$. 欲使 $|\overrightarrow{AD}|^2=17 = t^2-2t+1+9t^2+24t+16 = 10t^2+22t+17$

$\Rightarrow 10t^2+22t=0, \quad 5t^2+11t=0, \quad t(5t+11)=0, \quad t=0 \text{ or } -\frac{11}{5}$

$\rightarrow \overrightarrow{BC}=(-1,4) \text{ or } \left(-\frac{16}{5}, \frac{-13}{5}\right)$

Hence $C=(1,5)$ or $\left(\frac{-6}{5}, \frac{-8}{5}\right) \Rightarrow$ 取 $\left(\frac{-6}{5}, \frac{-8}{5}\right)$ #

(*)

6. $a_{42} = a_{41}-a_{40} = a_{40}-a_{39}-a_{40} = -a_{39} = -(a_{38}-a_{37}) = -(a_{37}-a_{36}-a_{37}) = a_{36} = a_{30} = a_{18} = -a_{15}$

$a_{28} = a_{16}$

$\Rightarrow \begin{cases} a_{15}=-3 \\ a_{16}=5 \end{cases}$ 已知 $a_{16}=a_{15}-a_{14} \rightarrow a_{14}=a_{15}-a_{16} = -3-5=-8$ #

7. $\sin 2x = 2\sin x\cos x$, $\cos 2x = 2\cos^2 x - 1$, 分子: $2\sin x\cos x - 2\cos^2 x + 1 - 1 = 2\cos x(\sin x - \cos x)$

原式: $\lim_{x\to\frac{\pi}{4}}(-2\cos x) = -2\times\frac{\sqrt{2}}{2} = -\sqrt{2}$ #

8. 討論 x 的 range $\rightarrow$ $\begin{cases} |x|>1,\ f(x)=\lim_{n\to\infty}\dfrac{x\cdot x^{2n}+ax^2+bx-5}{x^{2n}+2} = \lim_{n\to\infty}\dfrac{x+a\cdot\frac{1}{x^{2n}}\cdot x^2+b\cdot\frac{1}{x^{2n}}\cdot x-5\cdot\frac{1}{x^{2n}}}{1+2\cdot\frac{1}{x^{2n}}} = x \\ |x|<1,\ f(x)=\lim_{n\to\infty}\dfrac{x+a\cdot\frac{1}{x^{2n}}\cdot x^2+b\cdot\frac{1}{x^{2n}}\cdot x-5\cdot\frac{1}{x^{2n}}}{1+2\cdot\frac{1}{x^{2n}}} = \dfrac{ax^2+bx-5}{2} \end{cases}$

$x\to1^+,\ f(x)=1$
$x\to1^-,\ f(x)=\frac{1}{2}(a+b-5)$ $\rightarrow 2=a+b-5,\ \underline{a+b=7}$

$x\to-1^+,\ f(x)=\frac{1}{2}(a-b-5)$
$x\to-1^-,\ f(x)=-1$ $\rightarrow -2=a-b-5,\ \underline{a-b=3}$

$\begin{cases} a+b=7 \\ a-b=3 \end{cases} \rightarrow (a,b)=(5,2)$ #

9. $|1-a|=4$, $a=5$ or -3 | $|c-a|=4$, $c=13$ or 5 | $|a-b|=|b-c|$ $\Rightarrow$ b為a,c中點

$(a,b,c)=(5,9,13), (5,5,5), (-3,5,13), (-3,1,5)$

共4組 #

10、令三實根 = p, q, r → $\begin{cases} p+q+r=3 \\ pq+pr+qr=3a \\ pqr=b \end{cases}$ $\begin{array}{l} \to 1 \geq pqr \\ \to a \geq \sqrt[3]{(pqr)^2} \geq 1 \end{array}$

故 $(\alpha, \beta) = (1, 1)$

11、設切點 $(p, p^3-12p-20)$ → 切線斜率 $3p^2-12$ → 切線 $y-(p^3-12p-20) = (3p^2-12)(x-p)$ 過 $(3, a)$

$a-p^3+12p+20 = (3p^2-12)(3-p)$
$= 9p^2-3p^3-36+12p$ → $2p^3-9p^2+56+a=0$

let $g(p) = 2p^3-9p^2+56+a$
$g'(p) = 6p^2-18p \overset{let}{=} 0$, $p = 0$ or 3
→ $g(0) = (56+a)$
$g(3) = (29+a)$

而 $(a+56)(a+29) < 0$ → $-56 < a < -29$ #

12、令 $x' = 2x+3y+4$, $y' = 3x-7y-5$ → $\frac{|x'|}{9} + \frac{|y'|}{4} = 1$ → 面積 $= (\frac{1}{2} \times 9 \times 4) \times 4 = 72$

$\left\| \begin{matrix} 2 & 3 \\ 3 & -7 \end{matrix} \right\| = |-14-9| = 23$，(原)$\times 23 = 72$，原面積 $= \frac{72}{23}$ #

13、坐標化，令拋物線頂點為 $(0,0)$，$y = ax^2$ 過 $(2, 8)$ → $8 = 4a$，$a = 2$ → $y = 2x^2$
令切點 (a, b) → 切線斜率 $= 4a$ → $\overline{PO}$ 斜率 $= -\frac{1}{4a}$
$(a, b > 0)$ 已知 $\overline{PR} = (b-1)$，$\overline{PO} = 1$. ↳ $\overline{OR} = (b-1)(4a)$

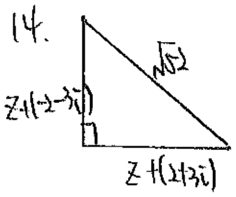
↓
$b = 2a^2$
$8b = 16a^2$

by 畢氏 → $(b-1)^2(16a^2) + (b-1)^2 = 1$ → $(b-1)^2(8b) + (b-1)^2 = 1$
$(b^2-2b+1)(8b) + b^2-2b = 0$ → $8b^3-16b^2+8b+b^2-2b = 8b^3-15b^2+6b = 0$，$b(8b^2-15b+6) = 0$

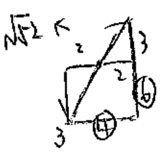
↓
$\frac{15 \pm \sqrt{225-192}}{16} = \frac{15 \pm \sqrt{33}}{16}$ $(b > 1)$

取 $b = \frac{15+\sqrt{33}}{16}$ #

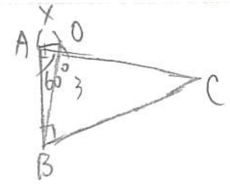
14.



$|z+(-2-3i)|^2 + |z+(2+3i)|^2 = 52$
$\Rightarrow 2|z|^2 + 2(13) = 52 \Rightarrow |z|^2 = 13$，$|z| = \sqrt{13}$ #



15.



$\overline{AB}=\sqrt{9-x^2} \rightarrow \overline{BC}=\sqrt{3}\cdot\sqrt{9-x^2}$

$2\int_0^3\left[\frac{1}{2}(\sqrt{9-x^2})(\sqrt{3})\sqrt{9-x^2}\right]dx=\sqrt{3}\int_0^3(9-x^2)dx=\sqrt{3}\cdot\left[9x-\frac{1}{3}x^3\right]\Big|_0^3=\sqrt{3}\cdot(27-9)$

$=18\sqrt{3}$ #

16. 坐標化 → 令 $D(0,0)$、$A(0,2)$、$C(2,0)$ → $\overline{AC}$方程式: $x+y=2$

$G(a,b)$、$F(a-b,a+b)\in x+y=2$

$(a>b>0)$ ↳ $2a=2$, $a=1$

$\overline{CG}^2=(2-a)^2+b^2=1+b^2$, $B(-2,0)$、$E(-b,a)\rightarrow\overline{BE}^2=(b-2)^2+a^2=(b-2)^2+1\Rightarrow\overline{BE}^2=3\overline{CG}^2$

$(b-2)^2+1=3+3b^2$

$\Rightarrow b^2-4b+5=3b^2+3\Rightarrow 2b^2+4b-2=0$, $b^2+2b-1=0$, $b=\frac{-2\pm\sqrt{4+4}}{2}=\frac{-2\pm2\sqrt{2}}{2}=-1\pm\sqrt{2}$ (b取 $-1+\sqrt{2}$)

正方形面積 $=(\sqrt{a^2+b^2})^2=a^2+b^2=1+(1+2-2\sqrt{2})=4-2\sqrt{2}$ #

~~17.~~ 令 $x'=\frac{x}{2}$ ($x=2x'$), $y'=y$

$\begin{pmatrix}\frac{1}{2}&0\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x'\\y'\end{pmatrix}$

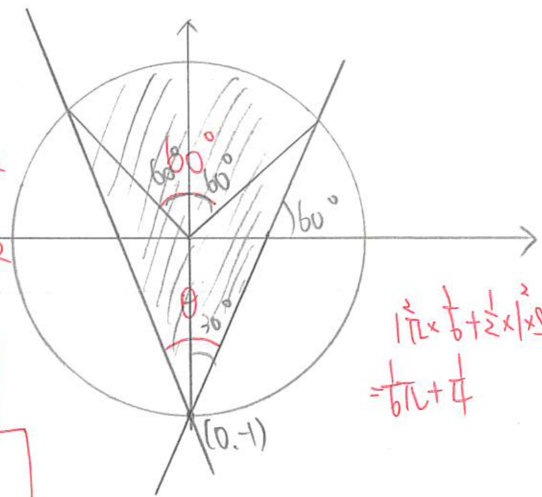
$\begin{vmatrix}\frac{1}{2}&0\\0&1\end{vmatrix}=\frac{1}{2}$ (面積倍率)

原式: $\begin{cases}(x')^2+(y')^2\le 1\\ y'+1\ge(\sqrt{3}+2)x' & m_1=\sqrt{3}+2\\ y'+1\ge-(\sqrt{3}+2)x' & m_2=-\sqrt{3}-2\end{cases}$

x'	0	$\frac{1}{\sqrt{3}+2}$
y'	-1	0

x'	0	$\frac{1}{-(\sqrt{3}+2)}$
y'	-1	0

~~斜率 $=\sqrt{3}=\tan\theta$, $\theta=60°$~~ $\sqrt{3}+2$



$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\frac{1}{\sqrt{3}}$、$\theta=30°$

$1^2\pi\times\frac{1}{6}+\frac{1}{2}\times1\times1^2\times\sin150°$

$=\frac{1}{6}\pi+\frac{1}{4}$

(原)$\times\frac{1}{2}=\frac{1}{6}\pi+\frac{1}{4}$

原 $=\frac{1}{3}\pi+\frac{1}{2}$ #

面積 $=1^2\pi\times\frac{120°}{360°}+(\frac{1}{2}\times1\times1\times\sin120°)\times2=\frac{1}{3}\pi+(\frac{\sqrt{3}}{2})$ #

18. $f(x)=\frac{x^2+2}{x}+\frac{64}{9}\left(\frac{x}{x^2-x+2}\right)=\frac{x^2-x+2}{x}+\frac{64}{9}\left(\frac{x}{x^2-x+2}\right)+1\ge2\times\sqrt{\frac{64}{9}}+1=2\times\frac{8}{3}+1=\frac{16+3}{3}=\frac{19}{3}$ #

(需check 等號成立… ok)

19. $\begin{cases} x_1 = \frac{1}{4}(x_2+x_4+6) \\ x_3 = \frac{1}{4}(x_2+x_6) \\ x_7 = \frac{1}{4}(x_4+x_8+2) \\ x_9 = \frac{1}{4}(x_6+x_8+6) \end{cases}$ $\begin{cases} x_2 = \frac{1}{4}(x_1+x_3+x_5-2) \\ x_4 = \frac{1}{4}(x_1+x_5+x_7) \\ x_6 = \frac{1}{4}(x_3+x_5+x_9-2) \\ x_8 = \frac{1}{4}(x_5+x_7+x_9) \end{cases}$

$\rightarrow x_1+x_3+x_7+x_9 = \frac{1}{4}(2x_2+2x_4+2x_6+x_8+(4)$

$= \frac{1}{2}(x_2+x_4+x_6+x_8)+\frac{7}{2}$

$x_2+x_4+x_6+x_8 = \frac{1}{4}(2x_1+2x_3+2x_7+x_9+4x_5-4)$

$x_2+x_4+x_6+x_8 = \frac{1}{2}(x_1+x_3+x_7+x_9)+x_5-1$

令 $x_2+x_4+x_6+x_8 = a \rightarrow x_1+x_3+x_7+x_9 = \frac{1}{2}a+\frac{7}{2}$

$a = \frac{1}{2}(\frac{1}{2}a+\frac{7}{2})+x_5-1$

$a = \frac{1}{4}a+\frac{7}{4}+x_5-1$, $x_5 = \frac{3}{4}a-\frac{3}{4}$

$x_5 = \frac{1}{4}(x_2+x_4+x_6+x_8) = \frac{1}{4}a = \frac{3}{4}a-\frac{3}{4} \rightarrow a = 3a-3$, $a=\frac{3}{2}$ 即 $x_5 = \frac{3}{4}\times\frac{3}{2}-\frac{3}{4} = \frac{9-6}{8} = \frac{3}{8}$ #

20. $x^2(-3-x)+y^2(-3-y)+z^2(-3-z) = -24 \Rightarrow -3(x^2+y^2+z^2)-(x^3+y^3+z^3) = -24$

$\Rightarrow 3(x^2+y^2+z^2)+(x^3+y^3+z^3) = 24$

$\begin{cases} x^2+y^2+z^2+2(xy+yz+xz) = 9 \Rightarrow x^2+y^2+z^2 = 9-2a \\ \frac{yz+xz+xy}{xyz} = -\frac{1}{3}, \text{令 } xy+xz+yz = a \rightarrow xyz = -3a \end{cases}$

已知 $x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-xz-yz)$

$x^3+y^3+z^3+9a = (-3)(9-2a-a) \Rightarrow x^3+y^3+z^3 = -27 \Rightarrow 3(x^2+y^2+z^2) = 51$

$= -3(9-3a)$

$= -27+9a$

$x^2+y^2+z^2 = 17$ #

二、計算題(共8分，每題8分)

1. $(\frac{a+b+c+d}{a}-1)(\frac{a+b+c+d}{b}-1)(\frac{a+b+c+d}{c}-1)(\frac{a+b+c+d}{d}-1) = (\frac{b+c+d}{a})(\frac{a+c+d}{b})(\frac{a+b+d}{c})(\frac{a+b+c}{d})$

$\frac{\frac{b}{a}+\frac{c}{a}+\frac{d}{a}}{3} \geq \sqrt[3]{\frac{bcd}{a^3}}$

$\geq 81\cdot\sqrt[3]{\frac{b^3a^3c^3d^3}{a^3b^3c^3d^3}} = 81\times1 = 81$ #

等號成立時，$(\frac{1}{a}-1) = (\frac{1}{b}-1) \Rightarrow a=b=c=d=\frac{1}{4}$

(極值存在)