7.使用不等式的縮放(n+1,n,n-1 展開)

$$\frac{2}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}} < \frac{2}{\sqrt{n}+\sqrt{n-1}}$$

$$\frac{2\left(\sqrt{n+1}-\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n+1}-\sqrt{n}\right)} < \frac{1}{\sqrt{n}} < \frac{2\left(\sqrt{n}-\sqrt{n-1}\right)}{\left(\sqrt{n}+\sqrt{n-1}\right)\left(\sqrt{n}-\sqrt{n-1}\right)}$$

$$2\left(\sqrt{n+1}-\sqrt{n}\right) < \frac{1}{\sqrt{n}} < 2\left(\sqrt{n}-\sqrt{n-1}\right)$$

$$2\left(\sqrt{2}-\sqrt{1}\right) < \frac{1}{\sqrt{1}} < 2\left(\sqrt{1}-\sqrt{0}\right)$$

$$2\left(\sqrt{3}-\sqrt{2}\right) < \frac{1}{\sqrt{2}} < 2\left(\sqrt{2}-\sqrt{1}\right)$$

$$2\left(\sqrt{4}-\sqrt{3}\right) < \frac{1}{\sqrt{3}} < 2\left(\sqrt{3}-\sqrt{2}\right)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$2\left(\sqrt{n+1}-\sqrt{n}\right) < \frac{1}{\sqrt{n}} < 2\left(\sqrt{n}-\sqrt{n-1}\right)$$

相加對消後 $2\left(\sqrt{n}-1\right) < a_n < 2\left(\sqrt{n+1}-1\right)$

$$\frac{2\left(\sqrt{n}-1\right)}{n} < \frac{a_n}{n} < \frac{2\left(\sqrt{n+1}-1\right)}{n}$$

$$\lim_{n\to\infty}\frac{2\left(\sqrt{n}-1\right)}{n} < \lim_{n\to\infty}\frac{a_n}{n} < \lim_{n\to\infty}\frac{2\left(\sqrt{n+1}-1\right)}{n}$$

又$\lim_{n\to\infty}\frac{2\left(\sqrt{n+1}-1\right)}{n} = \lim_{n\to\infty}\frac{2\left(\sqrt{n}-1\right)}{n} = 2$

由夾擠定理知$\lim_{n\to\infty}\frac{a_n}{n} = 2$