Inequalities from 2008 Mathematical Contests

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Chapter 1: Problems

Pro 1. ***(Vietnamese National Olympiad 2008)*** *Let x, y, z be distinct non-negative real numbers. Prove that*

$$\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}\geq\frac{4}{xy+yz+zx}.$$

∇

Pro 2. ***(Iranian National Olympiad (3rd Round) 2008)****. Find the smallest real K such that for each $x, y, z \in \mathbb{R}^+$:*

$$x\sqrt{y}+y\sqrt{z}+z\sqrt{x}\leq K\sqrt{(x+y)(y+z)(z+x)}$$

∇

Pro 3. ***(Iranian National Olympiad (3rd Round) 2008)****. Let $x, y, z \in \mathbb{R}^+$ and $x+y+z=3$. Prove that:*

$$\frac{x^3}{y^3+8}+\frac{y^3}{z^3+8}+\frac{z^3}{x^3+8}\geq\frac{1}{9}+\frac{2}{27}(xy+xz+yz)$$

∇

Pro 4. ***(Iran TST 2008.)*** *Let $a, b, c > 0$ and $ab+ac+bc=1$. Prove that:*

$$\sqrt{a^3+a}+\sqrt{b^3+b}+\sqrt{c^3+c}\geq 2\sqrt{a+b+c}$$

∇

Pro 5. ***Macedonian Mathematical Olympiad 2008.*** *Positive numbers a, b, c are such that $(a+b)(b+c)(c+a)=8$. Prove the inequality*

$$\frac{a+b+c}{3}\geq\sqrt[27]{\frac{a^3+b^3+c^3}{3}}$$

∇

Pro 6. ***(Mongolian TST 2008)*** *Find the maximum number C such that for any nonnegative x, y, z the inequality*

$$x^3+y^3+z^3+C(xy^2+yz^2+zx^2)\geq(C+1)(x^2y+y^2z+z^2x).$$

holds.

∇

Pro 7. ***(Federation of Bosnia, 1. Grades 2008.)*** *For arbitrary reals x, y and z prove the following inequality:*

$$x^2+y^2+z^2-xy-yz-zx \geq \max\{\frac{3(x-y)^2}{4}, \frac{3(y-z)^2}{4}, \frac{3(y-z)^2}{4}\}.$$

∇

Pro 8. ***(Federation of Bosnia, 1. Grades 2008.)*** *If a, b and c are positive reals such that $a^2+b^2+c^2=1$ prove the inequality:*

$$\frac{a^5+b^5}{ab(a+b)}+\frac{b^5+c^5}{bc(b+c)}+\frac{c^5+a^5}{ca(a+b)} \geq 3(ab+bc+ca)-2$$

∇

Pro 9. ***(Federation of Bosnia, 1. Grades 2008.)*** *If a, b and c are positive reals prove inequality:*

$$(1+\frac{4a}{b+c})(1+\frac{4b}{a+c})(1+\frac{4c}{a+b}) > 25$$

∇

Pro 10. ***(Croatian Team Selection Test 2008)*** *Let x, y, z be positive numbers. Find the minimum value of:*

$$(a) \quad \frac{x^2+y^2+z^2}{xy+yz}$$

$$(b) \quad \frac{x^2+y^2+2z^2}{xy+yz}$$

∇

Pro 11. ***(Moldova 2008 IMO-BMO Second TST Problem 2)*** *Let $a_1, \ldots, a_n$ be positive reals so that $a_1+a_2+\ldots+a_n \leq \frac{n}{2}$. Find the minimal value of*

$$A=\sqrt{a_1^2+\frac{1}{a_2^2}}+\sqrt{a_2^2+\frac{1}{a_3^2}}+\ldots+\sqrt{a_n^2+\frac{1}{a_1^2}}$$

∇

Pro 12. ***(RMO 2008, Grade 8, Problem 3)*** *Let $a, b \in [0, 1]$. Prove that*

$$\frac{1}{1+a+b} \leq 1 - \frac{a+b}{2} + \frac{ab}{3}.$$

∇

Pro 13. ***(Romanian TST 2 2008, Problem 1)*** *Let $n \geq 3$ be an odd integer. Determine the maximum value of*

$$\sqrt{|x_1 - x_2|} + \sqrt{|x_2 - x_3|} + \ldots + \sqrt{|x_{n-1} - x_n|} + \sqrt{|x_n - x_1|},$$

where x_i are positive real numbers from the interval $[0, 1]$

∇

Pro 14. ***(Romania Junior TST Day 3 Problem 2 2008)*** *Let a, b, c be positive reals with $ab + bc + ca = 3$. Prove that:*

$$\frac{1}{1+a^2(b+c)} + \frac{1}{1+b^2(a+c)} + \frac{1}{1+c^2(b+a)} \leq \frac{1}{abc}.$$

∇

Pro 15. ***(Romanian Junior TST Day 4 Problem 4 2008)*** *Determine the maximum possible real value of the number k, such that*

$$(a+b+c)\left(\frac{1}{a+b} + \frac{1}{c+b} + \frac{1}{a+c} - k\right) \geq k$$

for all real numbers $a, b, c \geq 0$ with $a + b + c = ab + bc + ca$.

∇

Pro 16. ***(Serbian National Olympiad 2008)*** *Let a, b, c be positive real numbers such that $x + y + z = 1$. Prove inequality:*

$$\frac{1}{yz + x + \frac{1}{x}} + \frac{1}{xz + y + \frac{1}{y}} + \frac{1}{xy + z + \frac{1}{z}} \leq \frac{27}{31}.$$

∇

Pro 17. ***(Canadian Mathematical Olympiad 2008)*** *Let a, b, c be positive real numbers for which $a + b + c = 1$. Prove that*

$$\frac{a-bc}{a+bc} + \frac{b-ca}{b+ca} + \frac{c-ab}{c+ab} \leq \frac{3}{2}.$$

∇

Pro 18. ***(German DEMO 2008)*** *Find the smallest constant C such that for all real x, y*

$$1 + (x+y)^2 \leq C \cdot (1+x^2) \cdot (1+y^2)$$

holds.

∇

Pro 19. ***(Irish Mathematical Olympiad 2008)*** *For positive real numbers a, b, c and d such that $a^2 + b^2 + c^2 + d^2 = 1$ prove that*

$$a^2b^2cd + +ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 \leq 3/32,$$

and determine the cases of equality.

∇

Pro 20. ***(Greek national mathematical olympiad 2008, P1)*** *For the positive integers $a_1, a_2, ..., a_n$ prove that*

$$\left(\frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i}\right)^{\frac{kn}{t}} \geq \prod_{i=1}^{n} a_i$$

where $k = max\{a_1, a_2, ..., a_n\}$ and $t = min\{a_1, a_2, ..., a_n\}$. When does the equality hold?

∇

Pro 21. ***(Greek national mathematical olympiad 2008, P2)***
If x, y, z are positive real numbers with $x, y, z < 2$ and $x^2 + y^2 + z^2 = 3$ prove that

$$\frac{3}{2} < \frac{1+y^2}{x+2} + \frac{1+z^2}{y+2} + \frac{1+x^2}{z+2} < 3$$

∇

Pro 22. ***(Moldova National Olympiad 2008)*** *Positive real numbers a, b, c satisfy inequality $a + b + c \leq \frac{3}{2}$. Find the smallest possible value for: $S = abc + \frac{1}{abc}$*

∇

Pro 23. ***(British MO 2008)*** *Find the minimum of $x^2+y^2+z^2$ where $x,y,z \in \mathbb{R}$ and satisfy $x^3+y^3+z^3-3xyz=1$*

∇

Pro 24. ***(Zhautykov Olympiad, Kazakhstan 2008, Question 6)*** *Let a,b,c be positive integers for which $abc=1$. Prove that*

$$\sum \frac{1}{b(a+b)} \geq \frac{3}{2}.$$

∇

Pro 25. ***(Ukraine National Olympiad 2008, P1)*** *Let x, y and z are non-negative numbers such that $x^2+y^2+z^2=3$. Prove that:*

$$\frac{x}{\sqrt{x^2+y+z}}+\frac{y}{\sqrt{x+y^2+z}}+\frac{z}{\sqrt{x+y+z^2}} \leq \sqrt{3}$$

∇

Pro 26. ***(Ukraine National Olympiad 2008, P2)*** *For positive a,b,c,d prove that*

$$(a+b)(b+c)(c+d)(d+a)(1+\sqrt[4]{abcd})^4 \geq 16abcd(1+a)(1+b)(1+c)(1+d)$$

∇

Pro 27. ***(Polish MO 2008, Pro 5)*** *Show that for all nonnegative real values an inequality occurs:*

$$4(\sqrt{a^3b^3}+\sqrt{b^3c^3}+\sqrt{c^3a^3}) \leq 4c^3+(a+b)^3.$$

∇

Pro 28. ***(Chinese TST 2008 P5)*** *For two given positive integers $m,n>1$, let $a_{ij}(i=1,2,\cdots,n, j=1,2,\cdots,m)$ be nonnegative real numbers, not all zero, find the maximum and the minimum values of f, where*

$$f=\frac{n\sum_{i=1}^{n}(\sum_{j=1}^{m}a_{ij})^2+m\sum_{j=1}^{m}(\sum_{i=1}^{n}a_{ij})^2}{(\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij})^2+mn\sum_{i=1}^{n}\sum_{i=j}^{m}a_{ij}^2}$$

∇

Pro 29. ***(Chinese TST 2008 P6)*** *Find the maximal constant M, such that for arbitrary integer $n\geq 3$, there exist two sequences of positive real number $a_1,a_2,\cdots,a_n$, and $b_1,b_2,\cdots,b_n$, satisfying*
(1):$\sum_{k=1}^{n}b_k=1, 2b_k\geq b_{k-1}+b_{k+1}, k=2,3,\cdots,n-1;$
(2):$a_k^2\leq 1+\sum_{i=1}^{k}a_ib_i, k=1,2,3,\cdots,n, a_n\equiv M.$