116 Problems in Algebra

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Contents

Function Equation Problems

1) Find all functions $f: N \to N$ such that:

$$f(2010f(n) + 1389) = 1 + 1389 + \cdots + 1389^{2010} + n \quad (\forall n \in \mathbb{N}).$$

(Proposed by Mohammad Jafari)

2) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y:

$$f(x + y) = f(x).f(y) + xy$$

(Proposed by Mohammad Jafari)

3) Find all functions $f: R - \{1\} \to R$ such that:

$$f(xy) = f(x)f(Y) + xy \qquad (\forall x, y \in \mathbb{R} - \{1\})$$

(Proposed by Mohammad Jafari)

4) Find all functions $f: Z \to Z$ such that:

$$f(x) = 2f\big(f(x)\big) \qquad (\forall x \in \mathbb{Z})$$

(Proposed by Mohammad Jafari)

5) Find all functions $f, g: \mathbb{Z} \to \mathbb{Z}$ such that:

$$f(x) = 3f\big(g(x)\big) \qquad (\forall x \in \mathbb{Z})$$

(Proposed by Mohammad Jafari)

6) Find all functions $f: \mathbb{Z} \to \mathbb{Z}$ such that:

$$7f(x) = 3f\big(f(x)\big) + 2x \qquad (\forall x \in \mathbb{Z})$$

(Proposed by Mohammad Jafari)

7) Find all functions $f: \mathbb{Q} \to \mathbb{Q}$ such that:

$$f\big(x + y + f(x + y)\big) = 2f(x) + 2f(y) \qquad (\forall x, y \in \mathbb{Q})$$

(Proposed by Mohammad Jafari)

8) For all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x + f(x) + y) = x + f(x) + 2f(y) \qquad (\forall x, y \in \mathbb{R})$$

Prove that $f(x)$ is a bijective function.

(Proposed by Mohammad Jafari)

9) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x + f(x) + 2y) = x + f\big(f(x)\big) + 2f(y) \qquad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

10) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x + f(x) + 2y) = x + f(x) + 2f(y) \qquad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

11) For all $f: \mathbb{R} \to \mathbb{R}$ such that :

$$f(x + f(x) + 2y) = x + f(x) + 2f(y) \qquad (\forall x, y \in \mathbb{R})$$

Prove that $f(0) = 0$.

(Proposed by Mohammad Jafari)

12) Find all functions $f: \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all real numbers $x > y > 0$:

$$f(x - y) = f(x) - f(x).f\left(\frac{1}{x}\right).y$$

(Proposed by Mohammad Jafari)

13) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f\Big(f\big(x + f(y)\big)\Big) = x + f(y) + f(x + y) \qquad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

14) Find all functions $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that:

$$f(f\big(x + f(y)\big)) = 2x + f(x + y) \qquad (\forall x, y \in \mathbb{R}^+ \cup \{0\})$$

(Proposed by Mohammad Jafari)

15) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x + f(x) + 2f(y)) = x + f(x) + y + f(y) \qquad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

16) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f\big(2x + 2f(y)\big) = x + f(x) + y + f(y) \quad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

17) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f\big(f(x) + 2f(y)\big) = f(x) + y + f(y) \quad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

18) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

i) $f\big(x^2 + f(y)\big) = f(x)^2 + f(y) \qquad (\forall x, y \in \mathbb{R})$

ii) $f(x) + f(-x) = 0 \qquad (\forall x \in \mathbb{R}^+)$

iii) The number of the elements of the set $\{x \mid f(x) = 0, x \in \mathbb{R}\}$ is finite.

(Proposed by Mohammad Jafari)

19) For all injective functions $f: \mathbb{R} \to \mathbb{R}$ such that:
$f\big(x + f(x)\big) = 2x \quad (\forall x \in \mathbb{R})$
Prove that $f(x) + x$ is bijective.

(Proposed by Mohammad Jafari)

20) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f\big(x + f(x) + 2f(y)\big) = 2x + y + f(y) \qquad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

21) For all functions $f, g, h: \mathbb{R} \to \mathbb{R}$ such that f is injective and h is bijective satisfying $f\big(g(x)\big) = h(x)$ $(\forall x \in \mathbb{R})$, prove that $g(x)$ is bijective function.

(Proposed by Mohammad Jafari)

22) Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$f(2x + 2f(y)) = x + f(x) + 2y \qquad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

23) Find all functions $f: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+$ such that:

$$f\left(\frac{x+f(x)}{2} + y\right) = f(x) + y \qquad (\forall x, y \in \mathbb{R}^+ \cup \{0\})$$

(Proposed by Mohammad Jafari)

24) Find all functions $f: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$ such that:

$$f\left(\frac{x+f(x)}{2} + f(y)\right) = f(x) + y \qquad (\forall x, y \in \mathbb{R}^+ \cup \{0\})$$

(Proposed by Mohammad Jafari)

25) For all functions $f: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ such that :

i) $f(x + y) = f(x) + f(y) \qquad (\forall x, y \in \mathbb{R}^+ \cup \{0\})$

ii) The number of the elements of the set $\left\{ x \mid f(x) = 0, x \in \mathbb{R}^+ \cup \{0\} \right\}$ is finite.

Prove that f is injective function.

(Proposed by Mohammad Jafari)

26) Find all functions $f: \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}$ such that:

i) $f(x + f(x) + 2y) = f(2x) + 2f(y) \qquad (\forall x, y \in \mathbb{R}^+ \cup \{0\})$

ii) The number of the elements of the set $\left\{ x \mid f(x) = 0, x \in \mathbb{R}^+ \cup \{0\} \right\}$ is finite.

(Proposed by Mohammad Jafari)

27) Find all functions $f: \mathbb{R} \rightarrow \mathbb{R}$such that:

i) $f(f(x) + y) = x + f(y) \qquad (\forall x, y \in \mathbb{R})$

ii) $\forall x \in \mathbb{R}^+; \exists y \in \mathbb{R}^+ \text{such that } f(y) = x$

(Proposed by Mohammad Jafari)

28) Find all functions $f: \mathbb{R}\{0\} \to \mathbb{R}$ such that:

i) $f(f(x)+y) = x + f(y) \quad (\forall x, y \in \mathbb{R})$

ii) The set $\{x \mid f(x) = -x, x \in \mathbb{R}\}$ has a finite number of elements.

(Proposed by Mohammad Jafari)

29) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f\big(f(f(x)) + f(y) + z\big) = x + f(y) + f\big(f(z)\big) \qquad (\forall x, y, z \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

30) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f\left(\frac{x+f(x)}{2} + y + f(2z)\right) = 2x - f(x) + f(y) + 2f(z) \qquad (\forall x, y, z \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

31) Find all functions $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that:

$$f\left(\frac{x+f(x)}{2} + y + f(2z)\right) = 2x - f(x) + f(y) + 2f(z) \quad (\forall x, y, z \in \mathbb{R}^+ \cup \{0\})$$

(Proposed by Mohammad Jafari)

32) (IRAN TST 2010) Find all non-decreasing functions $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that:

$$f\left(\frac{x+f(x)}{2} + y\right) = 2x - f(x) + f(f(y)) \quad (\forall x, y \in \mathbb{R}^+ \cup \{0\})$$

(Proposed by Mohammad Jafari)

33) Find all functions $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that:

$$f(x + f(x) + 2y) = 2x + f(2f(y)) \qquad (\forall x, y \in \mathbb{R}^+ \cup \{0\})$$

(Proposed by Mohammad Jafari)

34) Find all functions $f: \mathbb{Q} \to \mathbb{Q}$ such that:

$$f(x + f(x) + 2y) = 2x + 2f(f(y)) \qquad (\forall x, y \in \mathbb{Q})$$

(Proposed by Mohammad Jafari)

35) Find all functions $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that:

$$f\left(\frac{x+f(x)}{2} + y + f(2z)\right) = 2x - f(x) + F(f(y)) + 2f(z) \qquad (\forall x, y, z \in \mathbb{R}^+ \cup \{0\})$$

(Proposed by Mohammad Jafari)

36) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f\big(x - f(y)\big) = f(y)^2 - 2xf(y) + f(x) \qquad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

37) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$(x-y)\big(f(x) + f(y)\big) = (x+y)\big(f(x) - f(y)\big) \qquad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

38) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x-y)(x+y) = (x-y)(f(x) + f(y)) \qquad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

39) Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(x-y)(x+y) = f(x+y)(x-y) \qquad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

40) Find all non-decreasing functions $f, g: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that:

$g(x) = 2x - f(x)$

prove that f and g are continues functions.

(Proposed by Mohammad Jafari)

41) Find all functions $f: \{\, x \mid x \in \mathbb{Q}, x > 1 \,\} \to \mathbb{Q}$ such that :

$$f(x)^2 . f(x^2)^2 + f(2x) . f\left(\frac{x^2}{2}\right) = 1 \qquad \forall x \in \{x \mid x \in \mathbb{Q}, x > 1\}$$

(Proposed by Mohammad Jafari)

42) (IRAN TST 2011) Find all bijective functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f\big(x + f(x) + 2f(y)\big) = f(2x) + f(2y) \qquad (\forall x, y \in \mathbb{R})$$

(Proposed by Mohammad Jafari)

43) Find all functions $f: \mathbb{R}^+ \to \mathbb{R}^+$ such that:

$$f(x + f(x) + y) = f(2x) + f(y) \qquad (\forall x, y \in \mathbb{R}^+)$$

(Proposed by Mohammad Jafari)

44) Find all functions $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that:

$$f\big(x + f(x) + 2f(y)\big) = 2f(X) + y + f(y) \qquad (\forall x, y \in \mathbb{R}^+ \cup \{0\})$$

(Proposed by Mohammad Jafari)

45) Find all functions $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that:

i) $f\big(x + f(x) + f(2y)\big) = 2f(x) + y + f(y) \qquad (\forall x, y \in \mathbb{R}^+ \cup \{0\})$

ii) $f(0) = 0$

(Proposed by Mohammad Jafari)

46) Find all functions $f: \mathbb{R}^+ \to \mathbb{R}^+$ such that:

$$f(x + y^n + f(y)) = f(x) \qquad (\forall x, y \in \mathbb{R}^+,\ n \in \mathbb{N}, n \geq 2)$$

(Proposed by Mohammad Jafari)

47) Find all functions $f: \mathbb{N} \to \mathbb{N}$ such that:

$$f(n-1) + f(n+1) < 2f(n) \qquad (\forall n \in \mathbb{N}, n \geq 2)$$

(Proposed by Mohammad Jafari)

48) Find all functions $f: \{A \mid A \in \mathbb{Q}, A \geq 1\} \to \mathbb{Q}$ such that:

$$f(xy^2) = f(4x).f(y) + \frac{f(8x)}{f(2y)}$$

(Proposed by Mohammad Jafari)

Inequality Problems:

49) For all positive real numbers x, y, z such that $x+y+z=2$ prove that :

$$\frac{x}{x^4+y+z+1}+\frac{y}{y^4+z+x+1}+\frac{z}{z^4+x+y+1} \leq 1$$

(Proposed by Mohammad Jafari)

50) For all positive real numbers a, b, c such that $a+b+c=6$ prove that :

$$\sum \sqrt[n]{\frac{a}{(a+b)(a+c)}} \leqq \frac{3}{\sqrt[n]{abc}} \text{ , } n \in \mathbb{N}, n \geq 3$$

(Proposed by Mohammad Jafari)

51) For all real numbers $a, b, c \in (2,4)$ prove that:

$$\frac{2}{a+b^2+c^3}+\frac{2}{b+c^2+a^3}+\frac{2}{c+a^2+b^3} < \frac{3}{a+b+c}$$

(Proposed by Mohammad Jafari)

52) For all positive real numbers a, b, c prove that:

$$\frac{a}{a^4+a^2+1}+\frac{b}{b^4+b^2+1}+\frac{c}{c^4+c^2+1} < \frac{4}{3}$$

(Proposed by Mohammad Jafari)

53) For all real positive numbers a, b, c such that$1+\frac{1}{a+b+c} < \frac{1}{\sqrt{a^2+b^2+c^2}}+\frac{1}{\sqrt{ab+bc+ac}}$prove that:

$$ab+bc+ca < a+b+c$$

(Proposed by Mohammad Jafari)

54) For all real numbers a, b, c such that $a \geq b \geq 0 \geq c$ and $a+b+c < 0$ prove that :

$a^3+b^3+c^3+ab^2+bc^2+ca^2 \leq 2a^2b+2b^2c+2c^2a$

(Proposed by Mohammad Jafari)

55) For all real numbers $0 < x_1 < x_2 < \cdots < x_{1390} < \frac{\pi}{2}$ prove that :

$sin^3x_1+cos^3x_2$+...+$sin^3x_{1389}+cos^3x_{1390} < 695$

(Proposed by Mohammad Jafari)

56) For all $\alpha_i \in \left(0, \frac{\pi}{2}\right)$ $(i = 1,2, \dots, n)$ prove that :

$$\frac{n}{2} \geq \min\{\sum_{i=1}^{n} sin^3\alpha_i . cos^3\alpha_{i+1}, \sum_{i=1}^{n} sin^3\alpha_{i+1} . cos^3\alpha_i\} \qquad (\alpha_{n+1} = \alpha_1)$$

(Proposed by Mohammad Jafari)

57) For all real numbers $a, b, c \in [2,3]$ prove that:

$$\frac{1}{ab(2a-b)} + \frac{1}{bc(2b-c)} + \frac{1}{ca(2c-a)} \geq \frac{1}{9}$$

(Proposed by Mohammad Jafari)

58) For all real numbers $a, b, c \in [1,2]$ prove that:

$$\frac{2}{ab(3a-b)} + \frac{2}{bc(3b-c)} + \frac{2}{ca(3c-a)} \leq 3$$

(Proposed by Mohammad Jafari)

59) For all real positive numbers a, b, c prove that:

$$\frac{a}{3a+b+c} + \frac{b}{3b+c+a} + \frac{c}{3c+a+b} \leq \frac{3}{5}$$

(Proposed by Mohammad Jafari)

60) For all positive real numbers such that $abc = 1$ prove that:

$$\frac{a}{\sqrt{b^2+3}} + \frac{b}{\sqrt{c^2+3}} + \frac{c}{\sqrt{a^2+3}} \geq \frac{3}{2}$$

(Proposed by Mohammad Jafari)

61) For all positive real numbers a, b, c such that $a + b + c = 1$ prove that:

$$(\frac{ac}{\sqrt{a+b}} + \frac{ba}{\sqrt{b+c}} + \frac{cb}{\sqrt{c+a}})^2 + (\frac{bc}{\sqrt{a+b}} + \frac{ca}{\sqrt{c+b}} + \frac{ab}{\sqrt{c+a}})^2 \leq \frac{1}{3}$$

(Proposed by Mohammad Jafari)

62) For all positive real numbers a, b, c such that $a + b + c = 1$ prove that:

$$\frac{1}{\sqrt{a+b}} + \frac{1}{\sqrt{b+c}} + \frac{1}{\sqrt{c+a}} \leq \frac{1}{\sqrt{2abc}}$$

(Proposed by Mohammad Jafari)

63) For all positive real numbers a, b, c such that $ab + bc + ca = \frac{2}{3}$ prove that:

$$\frac{1}{\sqrt{a+b}} + \frac{1}{\sqrt{b+c}} + \frac{1}{\sqrt{c+a}} \leq \frac{1}{\sqrt{abc}}$$

(Proposed by Mohammad Jafari)

64) For all positive real numbers a, b, c such that $a + b + c = 3$prove that:

$$\frac{1}{a+b+c^2} + \frac{1}{b+c+a^2} + \frac{1}{c+a+b^2} \leq 1$$

(Proposed by Mohammad Jafari)

65) For all positive real numbers a, b, c prove that:

$$\frac{a^4+a^2+1}{a^6+a^3+1} + \frac{b^4+b^2+1}{b^6+b^3+1} + \frac{c^4+c^2+1}{c^6+c^3+1} \leq \frac{3}{a^2+a+1} + \frac{3}{b^2+b+1} + \frac{3}{c^2+c+1}$$

(Proposed by Mohammad Jafari)

66) For all positive real numbers a, b, c prove that:

$$\frac{a^4+a^2+1}{a^6+a^3+1} + \frac{b^4+b^2+1}{b^6+b^3+1} + \frac{c^4+c^2+1}{c^6+c^3+1} \leq 4$$

(Proposed by Mohammad Jafari)

67) For all positive real numbers a, b, c prove that:

$$\frac{1+b^2+c^4}{a+b^2+c^3} + \frac{1+c^2+a^4}{b+c^2+a^3} + \frac{1+a^2+b^4}{c+a^2+b^3} \geq 3$$

(Proposed by Mohammad Jafari)

68) For all real numbers $a, b, c \in (1,2)$ prove that:

$$\frac{4}{a+b+c} \geq \frac{1}{1+a+b^2} + \frac{1}{1+b+c^2} + \frac{1}{1+c+a^2}$$

(Proposed by Mohammad Jafari)

69) For all positive real numbers x, y, z such that $x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10$ prove that:

$$\frac{(x^2+1)^2}{4x} + \frac{(y^2+1)^2}{4y} + \frac{(z^2+1)^2}{4z} \geq x + y + z$$

(Proposed by Mohammad Jafari)

70) For all positive real numbers x, y, z such that $x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=10$ prove that:

$$4\sqrt{x+y+z} \geq x+y+z+3$$

(Proposed by Mohammad Jafari)

71) For all positive real numbers a, b, c prove that:

$$\left(\sum \frac{a^2}{b+c}\right)\left(\sum \frac{a}{(b+c)^2}\right) \geq \frac{9}{8}$$

(Proposed by Mohammad Jafari)

72) For all positive real numbers a, b, c prove that:

$$\left(\sum \frac{a}{(b+c)^2}\right)\left(\sum \frac{(b+c)^2}{a}\right) \geq \left(\sum \frac{2a}{b+c}\right)^2$$

(Proposed by Mohammad Jafari)

73) For all positive real numbers a, b, c prove that:

$$\left(\sum \frac{1}{a^2+b^2}\right)\left(\frac{a^3}{b^2+c^2}\right) \geq \left(\sum \frac{a}{b^2+c^2}\right)\left(\sum \frac{a^2}{b^2+c^2}\right)$$

(Proposed by Mohammad Jafari)

74) For all positive numbers a, b, c prove that :

$$2\left(\sum \frac{a}{b+c}\right)\left(\sum ab^2\right) \geq \left(\sum a\right)\left(\sum ab\right)$$

(Proposed by Mohammad Jafari)

75) For all $x_i \in \mathbb{N}$ $(i=1,2,\ldots,n)$ such that $x_i \neq x_j$,prove that:

$$\frac{1}{\frac{x_1}{1^2}}+\frac{2}{\frac{x_1}{1^2}+\frac{x_2}{2^2}}+\cdots+\frac{n}{\frac{x_1}{1^2}+\cdots+\frac{x_n}{n^2}} \leq \frac{n(n+3)}{2}$$

(Proposed by Mohammad Jafari)

76) For all $n \in \mathbb{N}$ $(n \geq 3)$ prove that :

$$n^{((n-1)!)^{\frac{1}{n-1}}}+n^{((n-2)!)^{\frac{1}{n-2}}}+\cdots+n^{(1)^{\frac{1}{1}}} < n^{\frac{n}{2}}+n^{\frac{n-1}{2}}+\cdots+n^1$$

(Proposed by Mohammad Jafari)

77) For all positive real numbers a, b, c such that $a + b + c = 1$ prove that:

$$\sum \frac{1}{(a+2b)(b+2c)} \geq 3$$

(Proposed by Mohammad Jafari)

78) For all positive real numbers a, b, csuch that $a + b + c = 2$ prove that:

$$\sum \frac{1}{(a^3+b+1)(1+b+c^3)} \leq 1$$

(Proposed by Mohammad Jafari)

79) For all positive real numbers a, b, c such that $\sqrt{a} + \sqrt{b} + \sqrt{c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ prove that:

$$\min\{\frac{a}{1+b^2}, \frac{b}{1+c^2}, \frac{c}{1+a^2}\} \leq \frac{1}{2}$$

(Proposed by Mohammad Jafari)

80) For all positive real numbers a, b, c such that $\frac{1}{a^m} + \frac{1}{b^m} + \frac{1}{c^m} = 3a^n b^n c^n$ $(m, n \in \mathbb{N})$ prove that:

$$\min\left\{\frac{a^k}{1+b^l}, \frac{b^k}{1+c^l}, \frac{c^k}{1+a^l}\right\} \leq \frac{1}{2} \qquad (k, l \in \mathbb{N})$$

(Proposed by Mohammad Jafari)

81) For all positive real numbers x, y prove that:

$$\sum \frac{x^n}{x^{n-1} + x^{n-2}y + \cdots + y^{n-1}} \geq \frac{x+y+z}{n} \qquad (n \in \mathbb{N})$$

(Proposed by Mohammad Jafari)

82) For all positive real numbers x, y, z prove that:

$$\sum \frac{x^{2^n}}{\left(x^{2^0} + y^{2^0}\right)\left(x^{2^1} + y^{2^1}\right) \dots \left(x^{2^{n-1}} + y^{2^{n-1}}\right)} \geq \frac{x+y+z}{2^n}$$

(Proposed by Mohammad Jafari)

83) For all positive real numbers x, y, z such that $\frac{x}{x^2+4} + \frac{y}{y^2+4} + \frac{z}{z^2+4} = \frac{1}{5}$ prove that:

$$\frac{x}{x+6} + \frac{y}{y+6} + \frac{z}{z+6} < 2.4$$

(Proposed by Mohammad Jafari)

84) Find minimum real number k such that for all real numbers a, b, c :

$$\sum \sqrt{2(a^2+1)(b^2+1)} + k \geq 2\sum a + \sum ab$$

(Proposed by Mohammad Jafari)

85) (IRAN TST 2011) Find minimum real number k such that for all real numbers a, b, c, d :

$$\sum \sqrt{(a^2+1)(b^2+1)(c^2+1)} + k \geq 2(ab+bc+cd+da+ac+bd)$$

(Proposed by Dr.Amir Jafari and Mohammad Jafari)

86) For all positive real numbers a, b, c prove that:

$$1 + \frac{\sum ab}{\sum a^2} \leq \sum \frac{(a+b)^2 + c^2}{2a^2 + 2b^2 + c^2} \leq 2 + \frac{\sum ab}{\sum a^2}$$

(Proposed by Mohammad Jafari)

87) For all real numbers $1 \leq a, b, c$ prove that:

$$\sum \frac{(a+b)^2 + c}{2a + 2b + c} \leq \sum a$$

(Proposed by Mohammad Jafari)

Polynomial Problems:

88) Find all polynomials $p(x)$ and $q(x)$ with real coefficients such that:

$$\sum_{i=0}^{2010}[p(x-i).p(x-i-1).(x-i-3)] \geq q(x).q(x-2010) \qquad (\forall x \in \mathbb{R})$$

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89) Find all polynomials $p(x)$ and $q(x)$ such that:

i) $p\big(q(x)\big) = q\big(p(x)\big) \qquad (\forall x \in \mathbb{R})$

ii) $p(x) \geq -x\,, q(x) \leq -x \qquad (\forall x \in \mathbb{R})$

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90) Find all polynomials $p(x)$ and $q(x)$ such that:

i) $\forall x \in \mathbb{R}\ :\ p(x) > q(x)$

ii) $\forall x \in \mathbb{R}\ :\ p(x).q(x-1) = p(x-1).q(x)$

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91) Find all polynomials $p(x)$ and $q(x)$ such that:

$$p^2(x) + q^2(x) = (3x - x^3).p(x).q(x) \qquad \forall x \in (0, \sqrt{3})$$

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92) The polynomial $p(x)$ is preserved with real and positive coefficients. If the sum of its coefficient's inverse equals 1, prove that :

$$p(1).p(x) \geq (\sum_{i=0}^{4} \sqrt[4]{x^i})^4$$

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93) The polynomial $p(x)$ is preserved with real and positive coefficients and with degrees of n. If the sum of its coefficient's inverse equals 1 prove that :

$$\sqrt{p(4)} + 1 \geq 2^{n+1}$$

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94) The polynomial $p(x)$ is increasing and the polynomial $q(x)$is decreasing such that:

$2p\big(q(x)\big) = p\big(p(x)\big) + q(x) \qquad \forall x \in \mathbb{R}$

Show that there is $x_0 \in \mathbb{R}$ such that:

$$p(x_0) = q(x_0) = x_0$$

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95) Find all polynomials $p(x)$ such that for the increasing function $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$

$$2p\big(f(x)\big) = f\big(p(x)\big) + f(x)\,, p(0) = 0$$

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96) Find all polynomials $p(x)$ such that, for all non zero real numbers x,y,z that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ $we\ have$:

$$\frac{1}{p(x)} + \frac{1}{p(y)} = \frac{1}{p(z)}$$

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97) $p(x)$ is an even polynomial ($p(x) = p(-x)$) such that $p(0) \neq 0$ .If we can write $p(x)$ as a multiplication of two polynomials with nonnegative coefficients, prove that those two polynomials would be even too.

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98) Find all polynomials $p(x)$ such that:

$$p(x+2)(x-2) + p(x-2)(x+2) = 2xp(x) \qquad (\ \forall x \in \mathbb{R})$$

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99) If for polynomials $p(x)$ and $q(x)$ that all their roots are real:

$$sign\big(p(x)\big) = sign(q(x))$$

Prove that there is polynomial $H(x)$ such that$p(x)q(x) = H(x)^2$.

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100) For polynomials $p(x)$ and $q(x)$: $[p(x^2+1)] = [q(x^2+1)]$

Prove that: $P(x) = q(x)$.

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101) For polynomials $P(x)$ and $q(x)$ with the degree of more than the degree of the polynomial $l(x)$, we have :

$$\left[\frac{p(x)}{l(x)}\right] = \left[\frac{q(x)}{l(x)}\right] \qquad (\forall x \in \{\, x \mid l(x) \neq 0, x \in \mathbb{R}^+\,\})$$

Prove that: $p(x) = q(x)$.

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102) The plain and desert which compete for breathing play the following game :

The desert choses 3 arbitrary numbers and the plain choses them as he wants as the coefficients of the polynomial $((\ldots x^2 + \cdots x + \cdots))$.If the two roots of this polynomial are irrational, then the desert would be the winner, else the plain is the winner. Which one has the win strategy?

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103) The two polynomials $p(x)$ and $q(x)$ have an amount in the interval $[n-1, n]$ $(n \in \mathbb{N})$ for$x \in [0,1]$.If p is non-increasing such that:$p\big(q(nx)\big) = nq(p(x))$,prove that there is $x_0 \in [0,1]$ such that $q(p(x_0) = x_0$.

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Other Problems

104) Solve the following system in real numbers :

$$\begin{cases} a^2 + b^2 = 2c \\ 1 + a^2 = 2ac \\ c^2 = ab \end{cases}$$

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105) Solve the following system in real numbers :

$$\begin{cases} ab = \dfrac{c^2}{1 + c^2} \\ bc = \dfrac{a^2}{1 + a^2} \\ ca = \dfrac{b^2}{1 + b^2} \end{cases}$$

(Proposed by Mohammad Jafari)

106) Solve the following system in positive real numbers : $(m, n \in \mathbb{N})$

$$\begin{cases} (2 + a^n)(2 + b^m) = 9 \\ (2 + a^m)(2 - b^n) = 3 \end{cases}$$

(Proposed by Mohammad Jafari)

107) Solve the following system in real numbers :

$$\begin{cases} xy^2 = y^4 - y + 1 \\ yz^2 = z^4 - z + 1 \\ zx^2 = x^4 - x + 1 \end{cases}$$

(Proposed by Mohammad Jafari)

108) Solve the following system in real numbers :

$$\begin{cases} xy = y^6 + y^4 + y^2 + 1 \\ yz^3 = z^6 + z^4 + z^2 + 1 \\ zx^5 = x^6 + x^4 + x^2 + 1 \end{cases}$$

(Proposed by Mohammad Jafari)

109) Solve the following system in real numbers :

$$\begin{cases} x^2.sin^2y + x^2 = siny.sinz \\ y^2.sin^2z + y^2 = sinz.sinx \\ z^2.sin^2x + z^2 = sinx.siny \end{cases}$$

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110) Solve the following system in real numbers :

$$\begin{cases} (a^2+1).(b^2+1).(c^2+1) = \dfrac{81}{8} \\ (a^4+a^2+1).(b^4+b^2+1).(c^4+c^2+1) = \dfrac{81}{8}(abc)^{\frac{3}{2}} \end{cases}$$

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111) Solve the following system in real numbers :

$$\begin{cases} a^2 + bc = b^2 + ca \\ b^2 + ca = c^2 + ab \\ c^2 + ab = a^2 + bc \end{cases}$$

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112) Solve the following system in positive real numbers :

$$\begin{cases} \left(a+\dfrac{1}{a}\right)\left(b+\dfrac{1}{b}\right) = 2(c+\dfrac{1}{c}) \\ \left(b+\dfrac{1}{b}\right)\left(c+\dfrac{1}{c}\right) = 2(a+\dfrac{1}{a}) \\ \left(c+\dfrac{1}{c}\right)\left(a+\dfrac{1}{a}\right) = 2(b+\dfrac{1}{b}) \end{cases}$$

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113) Solve the following equation for $x \in (0, \frac{\pi}{2})$:

$$\frac{2\sqrt{x}}{\pi}+\sqrt{sinx} + \sqrt{tanx} = \frac{1}{2\sqrt{x}} + \sqrt{cotx} + \sqrt{cosx}$$

(Proposed by Mohammad Jafari)

114) Find all functions $f, g: \mathbb{R}^+ \to \mathbb{R}^+$ in the following system :

$$\begin{cases} (f(x)+y-1)(g(y)+x-1) = (x+y)^2 & \forall x,y,z \in \mathbb{R}^+ \\ (-f(x)+y)(g(y)+x) = (x+y+1)(y-x-1) & \forall x,y,z \in \mathbb{R}^+ \end{cases}$$

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115) Solve the following system in real positive numbers :

$$\begin{cases} -a^4 + a^3 + a^2 = b + c + d \\ -b^4 + b^3 + b^2 = c + d + a \\ -c^4 + c^3 + c^2 = d + a + b \\ -d^4 + d^3 + d^2 = a + b + c \end{cases}$$

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116) Solve the following equation in real numbers :

$$\frac{xy}{2xy + z^2} + \frac{yz}{2yz + x^2} + \frac{zx}{2zx + y^2} = 1$$

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