2023 年亞太數學奧林匹亞競賽 初選考試 (一) 試題

考試時間：2022 年 10 月 30 日上午 10:00 ～ 12:00

說明：本試題共兩頁，分成兩部分：選填題與非選擇題。

作答方式：

- 選填題用 2B 鉛筆在「答案卡」上作答；更正時，應以橡皮擦擦拭，切勿使用修正液（帶）。
- 非選擇題用藍、黑色原子筆在「答案卷」上作答；更正時，可以使用修正液（帶）。
- 未依規定畫記答案卡，致機器掃描無法辨識答案，或未使用藍、黑色原子筆書寫答案卷，致評閱人員無法辨認答案者，其後果由考生自行承擔。
- 不得使用量角器、計算器及其他電子設備。
- 答案卷每人一張，不得要求增補。

第一部分：選填題

說明：本部分共有五題，每一題或小題的配分標於題前，答錯不倒扣，未完全答對不給分。

答案卡填答注意事項：答案的數字位數少於填答空格數時，請適當地在前面填入 0。

1. 給定坐標平面上的六個點：$(0,4)$, $(2,8)$, $(3,1)$, $(6,0)$, $(7,6)$, $(10,5)$。則包含此六點的凸多邊形中，最小可能面積為 ①②.③。
2. 青蛙在一張五乘五的方格紙最左下角的格子中。每一次，牠可以從牠目前所在位置往右跳一格、往上跳一格或往右上跳一格。則青蛙跳到最右上角格子的方法總數為④⑤⑥。
3. 設 m、n 為正整數，$mn < 2023$ 且 $|n^2 - mn - m^2| = 1$，則 mn 的最大可能值為⑦⑧⑨⑩。
4. 令 A 為 $\{1, 2, \cdots, 2022\}$ 的一個子集，滿足：對於 A 的任意兩個相異元素 $x > y$，$x+y$ 都不能被 $x-y$ 整除。則 A 元素個數的最大可能值為 ⑪⑫⑬。

5. 已知方程式 $x^2+ax+b=0$ 的兩根爲 α 和 β, 方程式 $x^2+bx+c=0$ 的兩根爲 $\frac{1}{\alpha}$ 和 β, 其中 a,b,c 爲實數且 $(b+1)(c+1)\neq 0$. 若 $(\frac{c}{b+1}+\frac{a}{c+1})^4$ 的最小可能值爲 $\frac{q}{p}$, 其中 p,q 爲互質的正整數，則 $(p,q)=$(⑭ ⑮ ⑯ , ⑰ ⑱ ⑲)。

第二部分：非選擇題

說明：每題 7 分，每題配分亦標於題前。答案必須寫在「答案卷」上，並標明題號，同時必須寫出演算過程或理由，否則將予扣分甚至零分。作答使用藍、黑色原子筆書寫，除幾何作圖外不得使用鉛筆。若因字跡潦草、未標示題號、標錯題號等原因，致評閱人員無法清楚辨識，其後果由考生自行承擔。

一、 證明：對於任何大於 1 的正整數 n，n 都無法整除 2^n-1。

二、 設 $\triangle ABC$ 爲直角三角形，$\angle B$ 爲直角。有一個圓心在 BC 邊上的圓 ω 與 AC 邊相切。由點 A 向圓 ω 引另一條切線 AT，其中 T 爲切點且不在 AC 邊上。設點 D 爲 AC 邊的中點，再設 BD 與 AT 交於點 M。試證：$MT=MB$。

2023 APMO Taiwan Preliminary Round 1

10:00–12:00, October 30, 2022

General instructions.

- There are 2 pages of problems, consisting of fill-in problems and non-multiple-choice problems.
- Use 2B pencils to answer fill-in problems on the designated card. Use erasers only to make corrections for these, do not use correction tape/fluid.
- Use pens in blue or black ink to answer non-multiple-choice problems on the designated sheet of paper. Correction tape/fluid may be used to make corrections for this part.
- Contestants are held responsible for the consequences from failing to follow the instructions above so that the machine cannot read the designated card, or the answers for non-multiple-choice problems are illegible.
- Protractors, calculators and other electronic devices are prohibited.
- One sheet of paper for the non-multiple-choice problems is given to each contestants. No more supply is offered.

Part 1. Fill-in problems

Instruction. There are FIVE problems in this Part. Each problem is worth 7 points. There is no penalty for wrong answers. No marks will be awarded for answers that are not completely correct.

If the number of digits for the answers is less than the number of designated spaces, fill in a proper number of 0's at the beginning of your answer.

1. Six points are given on the coordinate plane: $(0, 4)$, $(2, 8)$, $(3, 1)$, $(6, 0)$, $(7, 6)$, $(10, 5)$. A convex polygon contains all six points. Then the minimum possible area of this polygon is $\underline{①②.③}$.
2. There is a 5×5 chessboard, with a frog sitting on its lower-left corner cell. On each turn, the frog can either jump one cell upward, one cell to the right, or one cell in the upper-right direction. Then the frog has $\underline{④⑤⑥}$ different ways to jump to the cell in the upper-right corner of the chessboard.

3. Let m and n be positive integers such that $mn < 2023$ and $|n^2 - mn - m^2| = 1$. Then the maximum possible value for mn is $\underline{⑦⑧⑨⑩}$.

4. Let A be a subset of $\{1, 2, \cdots, 2022\}$ satisfying the following: for any two distinct elements $x > y$ of A, $x + y$ is not divisible by $x - y$. Then the maximum number of elements in A is $\underline{⑪ ⑫ ⑬}$.

5. It is known that the two roots of $x^2+ax+b=0$ are α and β, and the two roots of $x^2+bx+c=0$ are $\frac{1}{\alpha}$ and β, in which a, b, c are real numbers with $(b+1)(c+1) \neq 0$. If the minimum possible value of $(\frac{c}{b+1} + \frac{a}{c+1})^4$ is $\frac{q}{p}$, with p, q being positive integers prime to each other, then $(p, q) =$ $\underline{(⑭ ⑮ ⑯ , ⑰ ⑱ ⑲)}$.

Part 2. Non-multiple-choice problems

Instruction. There are TWO problems in this part. Each problem is worth 7 points. Answers should be written in blue or black ink, except for graphics that can be drawn by pencil. The problem number should be indicated clearly. The intermediate steps and reasons should be clearly stated, or penalty in deduction of points will be incurred.

I. Prove that: for any positive integer n greater than 1, n can never divide $2^n - 1$.

II. Let $\triangle ABC$ be a right triangle with $\angle B$ being its right angle. A circle ω centering at some point on side BC is tangent to side AC. The other tangent line from point A to circle ω is AT, where T is the tangent point not on side AC. Let D be the midpoint of side AC, and M be the intersection of BD and AT. Prove that $MT = MB$.

解答:

1. 45.0
2. 321
3. 1870
4. 674
5. (027,256)