108-全國高中教師聯招 試題解

1. 單選

解：

(*) $\sin 2\theta = 2\sin\theta\cos\theta$ ， $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

(A) $1+\cos 200^0 + i\sin 200^0 = 1 + (2\cos^2 100^0 - 1) + 2\sin 100^0\cos 100^0 i =$
$2\cos 100^0(\cos 100^0 + i\sin 100^0) = -2\cos 80^0(-\cos 80^0 + i\sin 80^0) =$
$2\cos 80^0(\cos 80^0 - i\sin 80^0) = 2\cos 80^0(\cos 280^0 + i\sin 280^0)$ ⇨ $Arz(\theta) = 280^0$ #

(B) $-1+\cos 200^0 + i\sin 200^0 = -1 + (1 - 2\sin^2 100^0) + 2\sin 100^0\cos 100^0 i =$
$2\sin 100^0(-\sin 100^0 + i\cos 100^0) = 2\sin 100^0(-\cos 10^0 - i\sin 10^0) =$
$2\sin 100^0(\cos 190^0 + i\sin 190^0)$ ⇨ $Arz(\theta) = 190^0$ #

(C) $1+\cos 200^0 - i\sin 200^0 = 1 + (2\cos^2 100^0 - 1) - 2\sin 100^0\cos 100^0 i =$
$2\cos 100^0(\cos 100^0 - i\sin 100^0) = -2\cos 80^0(-\cos 80^0 - i\sin 80^0) =$
$2\cos 80^0(\cos 80^0 + i\sin 80^0)$ ⇨ $Arz(\theta) = 80^0$ #

(D) $1-\cos 200^0 - i\sin 200^0 = 1 - (1 - 2\sin^2 100^0) - 2\sin 100^0\cos 100^0 i =$
$2\sin 100^0(\sin 100^0 - i\cos 100^0) = 2\sin 100^0(\cos 10^0 + i\sin 10^0)$ ⇨ $Arz(\theta) = 10^0$ #

2. 單選

解：

(*) $(x - \frac{2}{x})^n$

(1) $(x-\frac{2}{x})^n$ ⇨ $C_A^n(x)^{n-A}(-\frac{2}{x})^A = C_A^n(x)^{n-2A}(-2)^A$ ⇨ $n - 2A = -2$ (*)

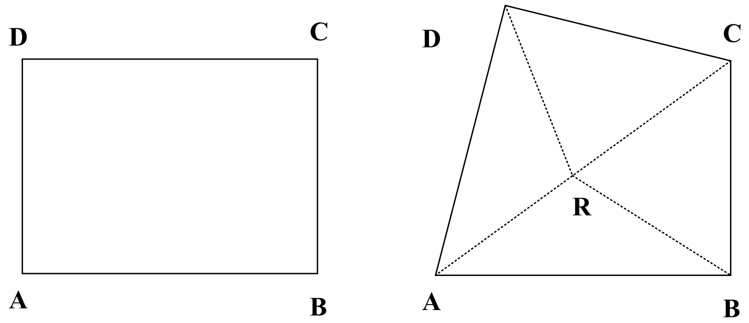
(2) $(x-\frac{2}{x})^n$ ⇨ $C_B^n(x)^{n-B}(-\frac{2}{x})^B = C_B^n(x)^{n-2B}(-2)^B$ ⇨ $n - 2B = -4$ (*)

⇨ $B - A = 1$ ⇨ $B = A + 1$ (***)

(3) $\dfrac{C_A^n(-2)^A}{C_B^n(-2)^B} = -1$ ⇨ $\dfrac{C_A^n(-2)^A}{C_{A+1}^n(-2)^{A+1}} = -1$ ⇨ $A = 5$ ， $B = 6$ ， $n = 8$ ###

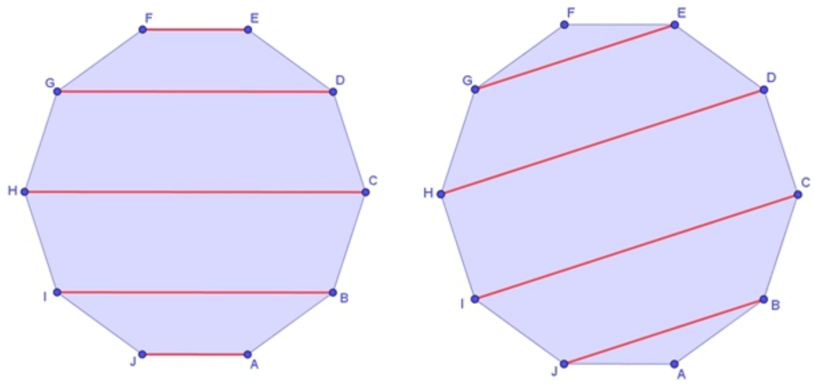
3. 單選

解：



(1) 以 $\overline{AR}$ 為半徑 ⇨ $V=\frac{4}{3}\pi R^3=\frac{4}{3}\pi(\frac{5}{2})^3=\frac{125}{6}\pi$ ###

4. 單選



Designed by ellipse 2019.05.12

(借圖)

解：

(1) $\frac{(C_2^5-2)\times 5+(C_2^4-2)\times 5}{C_4^{10}}=\frac{60}{210}=\frac{2}{7}$ ###

5. 單選

解：

(*) 設 t 為 $t^3+at^2+bt+c=0$ 三根，則 $t=16i$ ，$t=-1$ ，$t=81$

(1) 令 $t=x^4$ 代入 $t^3+at^2+bt+c=0$ ⇨ $x^{12}+ax^8+bx^4+c=0$ 則

(2) (i) $x^4=16i$ ⇨ $x=2(\cos(\frac{\pi}{8}+\frac{\pi k}{2})+i\sin(\frac{\pi}{8}+\frac{\pi k}{2}))$ (4 個虛根)

(ii) $x^4=-1$ ⇨ $x=\cos(\frac{\pi}{4}+\frac{\pi k}{2})+i\sin(\frac{\pi}{4}+\frac{\pi k}{2})$ (4 個虛根)

(iii) $x^4=81$ ⇨ $x=3(\cos(\frac{\pi k}{2})+i\sin(\frac{\pi k}{2}))$ (2 個虛根, 2 個實根) ###

6. 單選

解：

(1) $p(100萬) = p_1(100萬)\times 0 + p_1(1000元)\times\frac{1}{2} = \frac{1}{4}\times 0 + \frac{3}{4}\times\frac{1}{2} = \frac{3}{8}$ ###

7. 單選

解：

(1) E(x) = 0.6*[5 + E(x)] + 0.4*0.5*[5 + E(x)] + 0.4*0.5*10 ⇨ 解得 E(x) = 30 (借，利害了)

(1) 回到原地 $p = 0.8$，花 5 分鐘；走出森林 $q = 0.2$，花 10 分鐘

走出森林	第 **1** 次	第 **2** 次	第 **3** 次	第 **4** 次**….**	**…**
機 率	q	pq	p^2q	p^3q	**…**
所需時間	**10** 分鐘	**(5+10)**分鐘	(5×2+**10)**分鐘	(5×3+**10)**分鐘	**…**

$$E = q\times 10 + pq\times(5+10) + p^2q(5\times 2+10)+\ldots = q(10+15p+20p^2+25p^3\ldots)$$

⇨ $pE = q(10p+15p^2+20p^3+\ldots)$

⇨ $(1-p)E = q(10+5\times\frac{p}{1-p})$ ⇨ $E = 30$ ###

8. 單選

解：

(1) $f(x) = 1+x+x^2+x^3+x^4+\ldots+x^{n+1}+\ldots = \frac{1}{1-x}$

(2) $f'(x) = 1+2x+3x^2+4x^3+\ldots+(n+1)x^n+\ldots = (\frac{1}{1-x})' = (1-x)^{-2}$

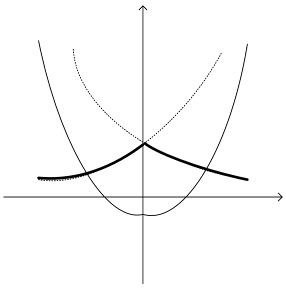
(3) $f''(x) = 2+3\cdot 2x+4\cdot 3x^2+\ldots+(n+1)\cdot nx^{n-1}+\ldots = (\frac{1}{1-x})'' = 2(1-x)^{-3}$

(4) $\frac{1}{2}f''(x) = 1+3x+6x^2+\ldots+\frac{(n+1)n}{2}x^{n-1}+\ldots = \frac{1}{(1-x)^3}$ ###

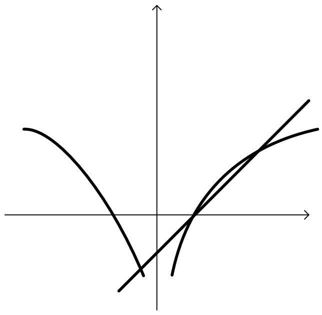
9. 複選題

解：

(A) $x^2-1=2^{-|x|}$ ⇨

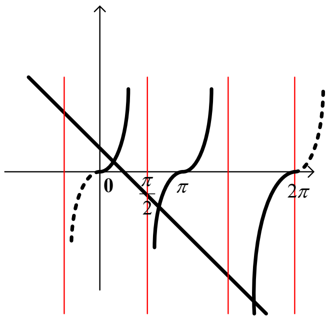


(B) $x-\log_2|x|=1$ ⇨ $x-1=\log_2|x|$ ⇨



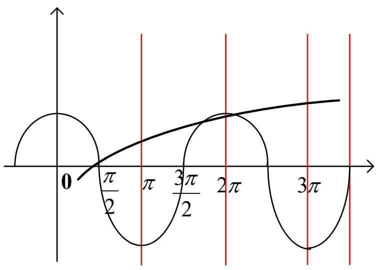
(√)

(C) $1-x=\tan x$ ⇨



(√)

(D) $\log_{10}x=\cos x$ ⇨



(√)

10. 複選題

解：

(1)

比賽場次	3	4	5
機 率	p^3+q^3	$C_2^3p^2q\times p+C_2^3pq^2\times q$	$C_2^4p^2q^2\times p+C_2^4p^2q^2\times q$

$\Rightarrow f(p)=3(p^3+q^3)+4(3p^3q+3pq^3)+5(6p^3q^2+6p^2q^3)$

$=3[(p+q)^3-3pq(p+q)]+12pq[(p+q)^2-2pq]+30p^2q^2(p+q)$

$=3[1-3pq]+12pq[1-2pq]+30p^2q^2$

$=3-9pq+12pq-24p^2q^2+30p^2q^2$

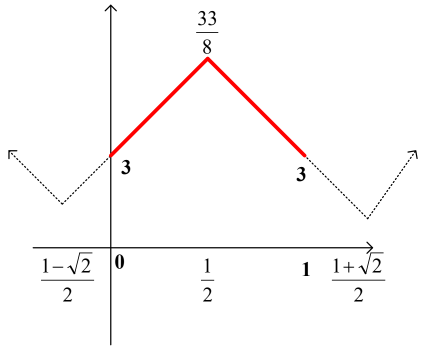
$=3+3pq+6p^2q^2$

$=6p^4-12p^3+3p^2+3p+3$

(2) $f(p)=6p^4-12p^3+3p^2+3p+3$

$\Rightarrow f'(p)=24p^3-36p^2+6p+3=3(8p^3-12p^2+2p+1)=0$

$\Rightarrow p=\dfrac{1}{2},\dfrac{1-\sqrt{2}}{2},\dfrac{1+\sqrt{2}}{2}$



$\Rightarrow$ ###

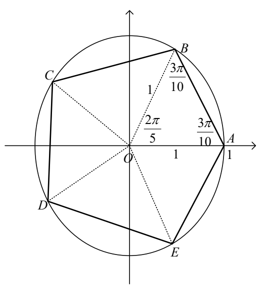
11. 複選題

解：

(A) $i\times z_1$ 代入 $z^5=(iz_1)^5=i^5z_1{}^5=iz_1{}^5=i$ （√）

(B) $|z_i-1|$表z_i到$A(1,0)$的距離

$\Rightarrow |z_1-1|=\overline{AB}>1$ 、$|z_2-1|=\overline{AC}>1$ 、$|z_3-1|=\overline{AD}>1$ 、$|z_4-1|=\overline{AE}>1$ （√）



(C) 設 $z^4+z^3+z^2+z+1=(z-z_1)(z-z_2)(z-z_3)(z-z_4)$，

令 $z=-1$ ⇨ $(1+z_1)(1+z_2)(1+z_3)(1+z_4)=1$ (√)

(D) 令 $x=2-z$ ⇨ $z=2-x$ 代入 $z^4+z^3+z^2+z+1=0$

⇨ $(2-x)^4+(2-x)^3+(2-x)^2+(2-x)+1=0$

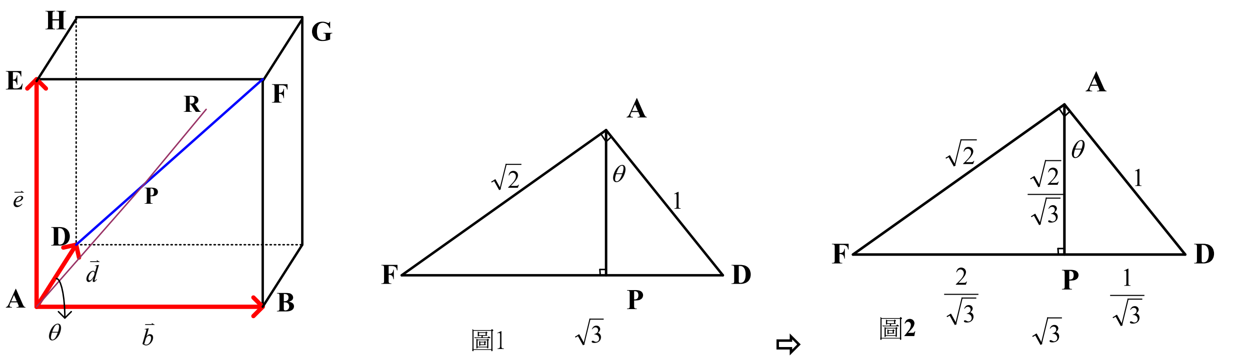
⇨ $x^4-9x^3+31x^2-49x+31=0$

⇨ $\begin{cases} x_1+x_2+x_3+x_4=9 \\ x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4=31 \\ \dots \end{cases}$

⇨ $x_1^2+x_2^2+x_3^2+x_4^2=81-62=19$ ###

12. 複選題

解：



(A)利用直角三角形 ADF(圖 1)相似關係得(圖 2)

⇨ **(向量)** $AP=\frac{2}{3}AD+\frac{1}{3}AF=\frac{2}{3}AD+\frac{1}{3}(AB+BF)=\frac{2}{3}\vec{d}+\frac{1}{3}(\vec{b}+\vec{e})=\frac{1}{3}(\vec{b}+2\vec{d}+\vec{e})$ (√)

(B)令(向量) $AR=tAP$ ⇨ $AR\cdot AD=|AR||AD|\cos\theta$ ⇨ $\frac{t}{3}(\vec{b}+2\vec{d}+\vec{e})\cdot\vec{d}=|AD|^2$ ⇨ $\frac{2t}{3}=1$

⇨ $t=\frac{3t}{2}$ ⇨ $AR=\frac{1}{2}(\vec{b}+2\vec{d}+\vec{e})$ (√)

(C) $\overline{DP}:\overline{PF}=1:2$

(D) $\cos\theta=\dfrac{\frac{\sqrt{2}}{\sqrt{3}}}{1}=\frac{\sqrt{6}}{3}$ (√)

1. 填充

解：

(1) $1+(1+2)+(1+2+3)+\ldots+(1+2+\ldots+n)=\sum_{k=1}^{n}\frac{k(k+1)}{2}=\frac{n(n+1)(n+2)}{6}$

(2) $\frac{1}{1}+\frac{1}{1+(1+2)}+\frac{1}{1+(1+2)+(1+2+3)}+\ldots+\frac{1}{\ldots+(1+2+\ldots+n)}$

$$=\sum_{k=1}^{n}\frac{1}{\frac{k(k+1)(k+2)}{6}}=\sum_{k=1}^{n}\frac{6}{k(k+1)(k+2)}$$

(3) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{6}{k(k+1)(k+2)}=6\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2}\left(\frac{1}{k(k+1)}-\frac{1}{(k+1)(k+2)}\right)=6\cdot\frac{1}{2}\left(\frac{1}{1\cdot 2}\right)=\frac{3}{2}$ ###

2. 填充

解：

填充2. 考慮 $g(x)=f(x)(x-1)+(-x-3)$

即 $g(\alpha)=f(\alpha)(\alpha-1)+(-\alpha-3)$

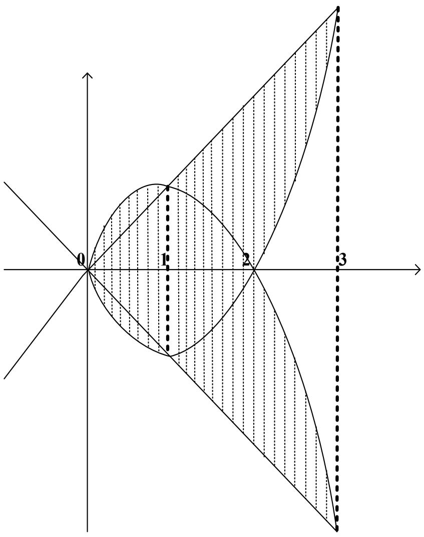
$g(\beta)=f(\beta)(\beta-1)+(-\beta-3)$ $\quad f(\alpha)=f(\beta)=f(\gamma)=0$

$g(\gamma)=f(\gamma)(\gamma-1)+(-\gamma-3)$,

$g(\alpha)g(\beta)g(\gamma)=(-\alpha-3)(-\beta-3)(-\gamma-3)=f(-3)=-58$

(借)

3. 填充



解：

(1) $\int_0^1 \pi(-x^2+2x)^2\,dx = \frac{8}{15}\pi$

(2) $\int_1^3 \pi(x)^2\,dx = \frac{26}{3}\pi$

(3) $\int_2^3 \pi(x^2-2x)^2\,dx = \frac{38}{15}\pi$

(4) ⇨ $\frac{8}{15}\pi + \frac{26}{3}\pi - \frac{38}{15}\pi = \frac{20}{3}\pi$ ###

4. 填充

解：

(1) 令 $x=\sqrt{108+\sqrt{108+\ldots}}$ ⇨ $x=\sqrt{108+x}$ ⇨ $x^2=108+x$ ⇨ $x=\frac{1\pm\sqrt{433}}{2}=10.9\ldots$###

5. 填充

解：

(1) 全列解法：

--A 抽到 **A** 情況→**ABCDE**

--B 抽到 **A** 情況→**BACDE**

--C 抽到 **A** 情況→**BCADE**、**CBADE**

--D 抽到 **A** 情況→**BCDAE**、**BDCAE**、**CBDAE**、**DBCAE**

--E 抽到 **A** 情況→**BCDEA**、**BCEDA**、**BDCEA**、**BECDA**、**CBDEA**、**CBEDA**、**DBCEA**、**EBCDA**

(2) 解法：

(i)若 **A** 取到自己的帽子，則 **BCDE** 只有一種取法，就是都正確的拿到自己的帽子。

(ii)若 **A** 沒有取到自己的帽子，則 **B,C,D,E** 之中至少有一個 **X**，所以共有 2^4-1 種。

(iii)合併以上兩種，共有 **15+1 = 16** 種情況。**(借)**

6. 填充

解：

(1) $\sum_{i=1}^{5} x_i = 10$，$\sum_{i=1}^{5} y_i = 400$，$\sum_{i=1}^{5} x_i^{\,2} = 30$ ⇨ $\mu_x = 2$，$\mu_y = 80$ #

(2) $\sum_{i=1}^{5}(x_i+\mu_x)^2 = \sum_{i=1}^{5} x_i^{\,2} + 2\mu_x\sum_{i=1}^{5} x_i + \sum_{i=1}^{5}\mu_x^{\,2} = 30+2\times2\times10+5\times2^2 = 90$ #

(3) $\dfrac{\sum_{i=1}^{5}(x_i+\mu_x)(y_i+\mu_y)}{\sum_{i=1}^{5}(x_i+\mu_x)^2}=\dfrac{311}{9}$ ⇨ $\sum_{i=1}^{5}(x_i+\mu_x)(y_i+\mu_y)=90\times\dfrac{311}{9}=3110$ #

⇨ $\sum_{i=1}^{5}(x_iy_i+x_i\mu_y+y_i\mu_x+\mu_x\mu_y)=3110$ ⇨ $\sum_{i=1}^{5}x_iy_i+\sum x_y\mu_y+\sum y_i\mu_x+\sum\mu_x\mu_y=3110$

⇨ $\sum_{i=1}^{5}x_iy_i+80\times10+2\times400+5\times2\times80=3110$ ⇨ $\sum_{i=1}^{5}x_iy_i+800+800+800=3110$

⇨ $\sum_{i=1}^{5}x_iy_i=710$ #

(4) $\sum_{i=1}^{5}(x_i-\mu_x)(y_i-\mu_y)=\sum_{i=1}^{5}(x_iy_i-x_i\mu_y-y_i\mu_x+\mu_x\mu_y)=$

$\sum_{i=1}^{5}x_iy_i-\sum x_y\mu_y-\sum y_i\mu_x+\sum\mu_x\mu_y=710-800-800+800=-90$ #

(5) $\sum_{i=1}^{5}(x_i-\mu_x)^2=\sum_{i=1}^{5}x_i{}^2-2\mu_x\sum_{i=1}^{5}x_i+\sum_{i=1}^{5}\mu_x{}^2=\ 30-2\times2\times10+5\times2^2=10$ #

(6) $\dfrac{\sum(x_i-\mu_x)(y_i-\mu_y)}{\sum(x_i-\mu_x)^2}=\dfrac{-90}{10}=-9$ #

(7) $y=ax+b$ ⇨ $\mu_y=a\mu_x+b$ ⇨ $80=-9\times2+b$ ⇨ $b=98$ ⇨ $y=-9x+98$ ###

7. 填充

解：

(1) $f'(0)=\lim_{x\to0}\dfrac{f(x)-f(0)}{x-0}=\lim_{x\to0}\dfrac{\dfrac{|x|}{1+|x|}-0}{x}$ 不存在 ###

(2) $f'(1)=\lim_{x\to1}\dfrac{f(x)-f(1)}{x-1}=\lim_{x\to1}\dfrac{\dfrac{|x|}{1+|x|}-\dfrac{1}{2}}{x-1}=\dfrac{1}{4}$ ###

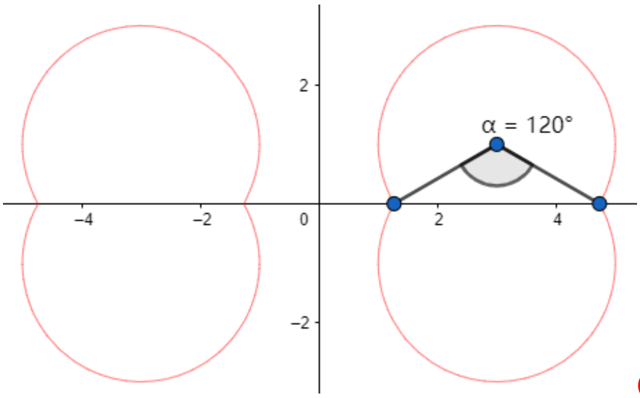
8. 填充

解：

(1) $\pi(2)^2 \times \frac{2}{3} = \frac{8}{3}\pi$

(2) $\frac{1}{2} \cdot 2 \cdot 2\sin 120^0 = \sqrt{3}$

(3) $(\frac{8}{3}\pi + \sqrt{3}) \times 4 \quad = \frac{32}{3}\pi + 4\sqrt{3}$ ###



(借)

9. 填充

填充第9題

$$a_{n+1} = 2a_n + 4^n$$

$$\frac{a_{n+1}}{2^n} = \frac{a_n}{2^{n-1}} + 2^n$$

$$\frac{a_2}{2^1} = \frac{a_1}{2^0} + 2^1$$

$$\frac{a_3}{2^2} = \frac{a_2}{2^1} + 2^2$$

:

:

$$\frac{a_n}{2^{n-1}} = \frac{a_{n-1}}{2^{n-2}} + 2^{n-1}$$

$$\frac{a_n}{2^{n-1}} = 3 + (2^1 + 2^2 + \cdots + 2^{n-1}) = 2^n + 1$$

$$a_n = \frac{2^n(2^n+1)}{2} = \frac{4^n + 2^n}{2}$$

解： **(借)**

解：

(1) 令$(a_{n+1} + k \cdot 4^n) = 2(a_n + k \cdot 4^{n-1})$ ⇨ $k = -2$ #

(2) $(a_{n+1} - 2 \cdot 4^n) = 2(a_n - 2 \cdot 4^{n-1})$

$(a_2 - 2 \cdot 4^1) = 2(a_1 - 2 \cdot 4^0)$

$(a_3 - 2\cdot 4^2) = 2(a_2 - 2\cdot 4^1)$

$(a_4 - 2\cdot 4^3) = 2(a_3 - 2\cdot 4^2)$

……

$(a_n - 2\cdot 4^{n-1}) = 2(a_{n-1} - 2\cdot 4^{n-2})$

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(3) $(a_n - 2\cdot 4^{n-1}) = 2^{n-1}(a_1 - 2\cdot 4^0)$ ⇨ $a_n = \frac{1}{2}(2^n + 4^n)$ ###

1. 計算

解：

(*) $\sum_{k=1}^{\infty} a_k$ 收斂 ⇨ $\lim_{k\to\infty} a_k = 0$

(1) $\sum_{k=1}^{\infty} a_k$ 收斂 ⇨ 令 $s_n = a_1 + a_2 + a_3 + \dots + a_n$ 且 $\lim_{n\to\infty} s_n = L$

⇨ $\lim_{k\to\infty} a_k = \lim_{k\to\infty}(s_k - s_{k-1}) = \lim_{k\to\infty} s_k - \lim_{k\to\infty} s_{k-1} = L - L = 0$ ###

(*) 「$\lim_{k\to\infty} a_k = 0$ ⇨ $\sum_{k=1}^{\infty} a_k$ 收斂 ？」

(1) 令 $s = \sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots$

⇨ $s = \frac{1}{1} + (\frac{1}{2}) + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + \dots + \frac{1}{16}) + \dots \geq \frac{1}{1} + (\frac{1}{2}) + 2(\frac{1}{4}) + 4(\frac{1}{8}) + \dots$

⇨ $s \geq \frac{1}{1} + (\frac{1}{2}) + \frac{1}{2} + \frac{1}{2} + \dots$ 發散 ###

2. 計算

解：

(*) $(1 + x + x^2)^n = a_0 + a_1 x + \dots + a_{2n} x^{2n}$

(1) 令 $x = 1$ ⇨ $(1 + 1 + 1^2)^n = a_0 + a_1 + \dots + a_{2n}$ ⇨ $3^n = a_0 + a_1 + \dots + a_{2n}$

(2) 令 $x=w$ ⇨ $(1+w+w^2)^n = a_0 + a_1 w + a_2 w^2 + a_3 w^3 + a_4 w^4 + \ldots + a_{2n} w^{2n}$

⇨ $0 = a_0 + a_1 w + a_2 w^2 + a_3 + a_4 w + \ldots + a_{2n} w^{2n}$

⇨ $0 = (a_0 + a_3 + a_6 + \ldots) + (a_1 + a_4 + a_7 + \ldots)w + (a_2 + a_5 + a_8 + \ldots)w^2$

⇨ $(a_0 + a_3 + a_6 + \ldots) = (a_1 + a_4 + a_7 + \ldots) = (a_2 + a_5 + a_8 + \ldots)$ **and** $3^n = a_0 + a_1 + \ldots + a_{2n}$

⇨ $\begin{cases} a + a + a + \ldots = 3^{n-1} \\ a + a + a + \ldots = 3^{n-1} \\ a + a + a + \ldots = 3^{n-1} \end{cases}$ ###

3. 計算

解：

(*) $\cos 2x = \cos x(\sin x + |\sin x|)$

(1) $0 \le x \le \pi$ ⇨ $\cos 2x = \cos x(\sin x + \sin x)$ ⇨ $\cos 2x = \sin 2x$

⇨ $\tan 2x = 1$ ⇨ $2x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$ ⇨ $x = \dfrac{\pi}{8}, \dfrac{5\pi}{8}$ ###

(2) $\pi \le x < 2\pi$ ⇨ $\cos 2x = \cos x(\sin x - \sin x)$ ⇨ $\cos 2x = 0$

⇨ $2x = \dfrac{5\pi}{2}, \dfrac{7\pi}{2}$ ⇨ $x = \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ ###