Some Own Problems In Number Theory

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Here are some problems composed by me.And the problems or their solutions have not been approved by someone else.So if any fault occurs,I shall take the whole responsibility.In this case,please inform me.Among the problems,many were posted by me in **AoPS**.So,I thank the mathlinkers who psted replies and solutions there.A notable fact is, I put the problems not in order to difficulty,just randomly-which I thought to be interesting.

1 Notations

I have used notaions which are used as usual.If not stated in a problem,then the variables are to be assumed positive integers.Otherwise they are stated.Here the notations are:
† $N = \{1, 2, ..., n, ...\}$ →the set of all natural numbers or positive integers or positive whole numbers.
† $Z = \{......, -2, -1, 0, 1, 2, ...\}$ →the set of all integers(all positive,negative,including non-negative 0).
$N_0 = \{0, 1, 2, ...\}$ →the set of all non-negative integers.
† $a \in A \rightarrow a$ is an element of A.
† p →prime.
† $a|b \rightarrow a$ divides b,i.e. b gives remainder 0 upon division by a.There is another notation on this-$b \vdots a$ means b is divisible by a.But we shall use the widely used first notation here.
† $a \not| b \rightarrow a$ does not divide b.
† $a|b \wedge c \rightarrow a$ divides both a and c.
† $gcd(a, b) = g \rightarrow g$ is the greatest common divisor of a and b.In other words,g is the largest positive integer such that $g|a \wedge b$.
† $\varphi(m)$ →the number of positive integers less or equal to m and co-prime to m.It is called **Euler's Totient Function** or **Euler's Phi Function**,shortly **phi** of m.
† $\forall n, \exists k$:→for all n,there exists k such that.
† $\lfloor x \rfloor$ →greatest integer function.$\lfloor x \rfloor$ is the largest integer less or equal to x.

2 Problems

Problem 1:
Prove that there exist no $(n, m) \in N$ so that $n + 3m$ and $n^2 + 3m^2$ both are perfect cubes.Find all such (m, n) if $(m, n) \in Z$.
Problem 2:
Find all primes p such that $11^p + 10^p$ is a perfect power.(A positive integer is called **perfect power** if it can be expressed as m^k for some natural $k > 1$.).
Problem 3:
In a single person game,Alex plays maintaining the following rules:
She is asked to consider the set of all natural numbers less than n on a board.Then she starts from 1 and whenever she gets an integer co-prime to n,she writes 1 on the board,otherwise she writes 0.That is she will write a binary sequence with either 1 or 0.
She denotes the number of $1'$s in this binary sequence of n by $\Phi_1(n)$ and the number of $0'$s by $\Phi_0(n)$.
Now,she wins if she can choose an n having at least 2 prime factors in the first choice such that $\Phi_1(n)|n$.Prove the following:
$\star$ 1 :There exist infinitely many n such that she can win in the first move.
$\star$ 2 :If she chooses an n having more than 3 prime factors,she can't never win.
$\star$ 3 :If $n = \prod_{i=1}^{n} p_i^{a^i}$,then $\prod_{i=1}^{n} p_i^{a_i - 1} | \Phi_0(n)$.
$\star$ 4 :Find all such n such that she can win.
Problem 4:
Let $F_n = 2^{2^n} + 1$ be the $n - th$ **Fermat number**.Prove that $2^{2^m + 2^n} | F_n^{F_m - 1} - 1 \forall_{m,n}$.
Problem 5:
Prove that for $a > 2, a^{a-1} - 1$ is never square-free.A number is called **square-free** if it has no square factor i.e. for no x, it is divisible by x^2.
Problem 6:
Show that $\frac{a^5+b^5}{a^3b^3+1}$ is a perfect cube for an infinite (a, b) whenever it is an integer.
Problem 7:
Prove that for all odd $p \not| c, ord_{p^k}(c) = ord_p(c).p^{k-1}$.If x is the smallest integer such that $a^x \equiv 1 \pmod m$,then x is called the **order** of a modulo m.And we write it,$ord_m(a) = x$.
Problem 8:
Show that for all prime $p \equiv 2 \pmod 3$,there exists a complete set of residue class of p such that the sum of its elements is divisible by p^2.
Problem 9:
For all $n \in N_0$,prove that $81 | 10^{n+1} - 10 - 9n$.
Problem 10:
Find all n such that $n | 2^{n!} - 1$.
Problem 11:
Find all n such that **(a).**$n | 2^n + 1$, **(b).**n| $3^n + 1$.
Problem 12:
A number is called a **perfect number** if the sum of its proper divisors(i.e. divisors less than the original number) is equal to the initial number.Determine

all perfect numbers having p factors(if there exists).
Problem 13:
Prove that,a number having only one prime factor can't be perfect.
Problem 14:
Find all (a, b) such that $ab|a^3 + b^3$.
Problem 15:
Solve in positive integers:$a^7 + b^7 = 823543(ac)^{1995}$.
Problem 16:
Find all n such that **(a).**$n^2 - 27n + 182$,**(b).**$n^2 - 27n + 183$ is a perfect square.
Problem 17:
Find all $(a, b) \in N_0$ such that $7^a + 11^b$ is a perfect square.
Problem 18:
Consider a complete set of residues modulo p.show that we can partition this set into two subsets with equal number of elements such that the sum of elements in each set is divisible by p.
Problem 19:
Let a_i, m be positive integers such that $a_i + m$ is a prime for all $1 \le i \le n$.Take the number N such that $N = \prod_{i=1}^{n} p_i^{a_i}$.Let S be the number of ways to express N as a product of m positive integers.Prove that $m^n|S$.
Problem 20:
Prove that $\forall n, \exists k : \frac{n}{\lfloor \sqrt[m]{n} \rfloor} > \frac{n+k}{\lfloor \sqrt[m]{n+k} \rfloor}$.
Problem 21:
Find all $(a, b, c, d) \in Z$ such that $abc - d = 1, bcd - a = 2$.

3 References

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