Problems of Vasc and Arqady

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1. Suppose that a, b, c are positive real numbers, prove that

$$1 < \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{b^2+c^2}} + \frac{c}{\sqrt{c^2+a^2}} \le \frac{3\sqrt{2}}{2}$$

2. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$2\left(\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b}\right) + (a+b+c)^2 \ge 4(a^2+b^2+c^2);$$

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} \le \frac{3(a^2+b^2+c^2)}{2(a+b+c)}.$$

3. For all reals a, b and c prove that:

$$\sum_{cyc}(a-b)(a-c)\sum_{cyc}a^2(a-b)(a-c) \ge \left(\sum_{cyc}a(a-b)(a-c)\right)^2$$

4. Let a, b and c are non-negatives such that $a+b+c+ab+ac+bc=6$. Prove that:

$$4(a+b+c)+abc \ge 13$$

5. Let a, b and c are non-negatives. Prove that:

$$(a^2+b^2-2c^2)\sqrt{c^2+ab} + (a^2+c^2-2b^2)\sqrt{b^2+ac} + (b^2+c^2-2a^2)\sqrt{a^2+bc} \le 0$$

6. If a, b, c are nonnegative real numbers, then

$$\sum_{cyc} a\sqrt{3a^2+5(ab+bc+ca)} \ge \sqrt{2}(a+b+c)^2;$$

$$\sum_{cyc} a\sqrt{2a(a+b+c)+3bc} \ge (a+b+c)^2;$$

$$\sum_{cyc} a\sqrt{5a^2+9bc+11a(b+c)} \ge 2(a+b+c)^2.$$

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7. If a, b, c are nonnegative real numbers, then

$$\sum_{cyc} a\sqrt{2(a^2+b^2+c^2)+3bc} \ge (a+b+c)^2;$$

$$\sum_{cyc} a\sqrt{4a^2+5bc} \ge (a+b+c)^2.$$

8. If a, b, c are nonnegative real numbers, then

$$\sum_{cyc} a\sqrt{ab+2bc+ca} \ge 2(ab+bc+ca);$$

$$\sum_{cyc} a\sqrt{a^2+4b^2+4c^2} \ge (a+b+c)^2.$$

9. If a, b, c are nonnegative real numbers, then

$$\sum \sqrt{a(b+c)(a^2+bc)} \ge 2(ab+bc+ca)$$

10. If a, b, c are positive real numbers such that $ab+bc+ca=3$, then

$$\sum_{cyc} \sqrt{a(a+b)(a+c)} \ge 6;$$

$$\sum_{cyc} \sqrt{a(4a+5b)(4a+5c)} \ge 27.$$

11. If a, b, c are nonnegative real numbers, then

$$\sum_{cyc} a\sqrt{(a+b)(a+c)} \ge 2(ab+bc+ca);$$

$$\sum_{cyc} a\sqrt{(a+2b)(a+2c)} \ge 3(ab+bc+ca);$$

$$\sum_{cyc} a\sqrt{(a+3b)(a+3c)} \ge 4(ab+bc+ca).$$

12. If a, b, c are nonnegative real numbers, then

$$\sum_{cyc} a\sqrt{(2a+b)(2a+c)} \ge (a+b+c)^2;$$

$$\sum_{cyc} a\sqrt{(a+b)(a+c)} \ge \frac{2}{3}(a+b+c)^2;$$

$$\sum_{cyc} a\sqrt{(4a+5b)(4a+5c)} \ge 3(a+b+c)^2.$$

13. If a, b, c are positive real numbers, then

$$a^3 + b^3 + c^3 + abc + 8 \geq 4(a + b + c);$$

$$a^3 + b^3 + c^3 + 3abc + 12 \geq 6(a + b + c);$$

$$4(a^3 + b^3 + c^3) + 15abc + 54 \geq 27(a + b + c).$$

14. If a, b, c are positive real numbers, then

$$\frac{a}{\sqrt{4a^2 + 3ab + 2b^2}} + \frac{b}{\sqrt{4b^2 + 3bc + 2c^2}} + \frac{c}{\sqrt{4c^2 + 3ca + 2a^2}} \leq 1;$$

$$\frac{a}{\sqrt{4a^2 + 2ab + 3b^2}} + \frac{b}{\sqrt{4b^2 + 2bc + 3c^2}} + \frac{c}{\sqrt{4c^2 + 2ca + 3a^2}} \leq 1;$$

$$\frac{a}{\sqrt{4a^2 + ab + 4b^2}} + \frac{b}{\sqrt{4b^2 + bc + 4c^2}} + \frac{c}{\sqrt{4c^2 + ca + 4a^2}} \leq 1.$$

The last is a known inequality.

15. If a, b, c are positive real numbers, then

$$1 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 2\sqrt{1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c}}.$$

16. Let x, y, z be real numbers such that $x + y + z = 0$. Find the maximum value of

$$E = \frac{yz}{x^2} + \frac{zx}{y^2} + \frac{xy}{z^2}.$$

17. If $a.b.c$ are distinct real numbers, then

$$\frac{ab}{(a-b)^2} + \frac{bc}{(b-c)^2} + \frac{ca}{(c-a)^2} + \frac{1}{4} \geq 0.$$

18. If a and b are nonnegative real numbers such that $a + b = 2$, then

$$a^a b^b + ab \geq 2;$$

$$a^a b^b + 3ab \leq 4;$$

$$a^b b^a + 2 \geq 3ab.$$

19. Let a, b, c, d and k be positive real numbers such that

$$(a + b + c + d)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}) = k$$

Find the range of k such that any three of a, b, c, d are triangle side-lengths.

20. If a, b, c, d, e are positive real numbers such that $a+b+c+d+e=5$, then

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+\frac{1}{e^2}+9 \geq \frac{14}{5}(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}).$$

21. Let a, b and c are non-negatives such that $ab+ac+bc=3$. Prove that:

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{1}{2}abc \leq 2.$$

22. Let a, b, c and d are positive numbers such that $a^4+b^4+c^4+d^4=4$. Prove that:

$$\frac{a^3}{bc}+\frac{b^3}{cd}+\frac{c^3}{da}+\frac{d^3}{ab} \geq 4.$$

23. Let $a \geq b \geq c \geq 0$ Prove that:

$$(a-b)^5+(b-c)^5+(c-a)^5 \leq 0;$$

$$\sum_{cyc}(5a^2+11ab+5b^2)(a-b)^5 \leq 0.$$

24. Let a, b and c are positive numbers. Prove that:

$$\frac{a}{a^2+bc}+\frac{b}{b^2+ac}+\frac{c}{c^2+ab} \leq \frac{3}{2\sqrt[3]{abc}}.$$

25. Let a, b and c are positive numbers. Prove that:

$$\sqrt{\frac{a+b}{c}}+\sqrt{\frac{b+c}{a}}+\sqrt{\frac{c+a}{b}} \geq \sqrt{\frac{11(a+b+c)}{\sqrt[3]{abc}}-15}.$$

26. Let a, b and c are non-negative numbers. Prove that:

$$9a^2b^2c^2+a^2b^2+a^2c^2+b^2c^2-4(ab+ac+bc)+2(a+b+c) \geq 0.$$

27. Let a, b, c, d be nonnegative real numbers such that $a \geq b \geq c \geq d$ and

$$3(a^2+b^2+c^2+d^2)=(a+b+c+d)^2.$$

Prove that

$$a \leq 3b;$$

$$a \leq 4c;$$

$$b \leq (2+\sqrt{3})c.$$

28. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{bc}{2a^2+bc}+\frac{ca}{2b^2+ca}+\frac{ab}{2c^2+ab}\le\frac{a^2+b^2+c^2}{ab+bc+ca};$$

$$\frac{2bc}{a^2+2bc}+\frac{2ca}{b^2+2ca}+\frac{2ab}{c^2+2ab}+\frac{a^2+b^2+c^2}{ab+bc+ca}\ge 3.$$

29. Let $a_1,a_2,...,a_n$ be real numbers sucht that

$$a_1,a_2,...,a_n\ge n-1-\sqrt{1+(n-1)^2},\quad a_1+a_2+...+a_n=n.$$

Prove that

$$\frac{1}{a_1^2+1}+\frac{1}{a_2^2+1}+...+\frac{1}{a_n^2+1}\ge\frac{n}{2}.$$

30. Let a,b,c be nonnegative real numbers such that $a+b+c=3$. For given real $p\neq -2$, find q such that the inequality holds

$$\frac{1}{a^2+pa+q}+\frac{1}{b^2+pb+q}+\frac{1}{c^2+pc+q}\le\frac{3}{1+p+q},$$

with two equality cases.
Some particular cases:

$$\frac{1}{a^2+2a+15}+\frac{1}{b^2+2b+15}+\frac{1}{c^2+2c+15}\le\frac{1}{6};$$

with equality for $a=0$ and $b=c=\frac{3}{2}$;

$$\frac{1}{8a^2+8a+65}+\frac{1}{8b^2+8b+65}+\frac{1}{8c^2+8c+65}\le\frac{1}{27};$$

with equality for $a=\frac{5}{2}$ and $b=c=\frac{1}{4}$;

$$\frac{1}{8a^2-8a+9}+\frac{1}{8b^2-8b+9}+\frac{1}{8c^2-8c+9}\le\frac{1}{3};$$

with equality for $a=\frac{3}{2}$ and $b=c=\frac{3}{4}$;

$$\frac{1}{8a^2-24a+25}+\frac{1}{8b^2-24b+25}+\frac{1}{8c^2-24c+25}\le\frac{1}{3};$$

with equality for $a=\frac{1}{2}$ and $b=c=\frac{5}{4}$;

$$\frac{1}{2a^2-8a+15}+\frac{1}{2b^2-8b+15}+\frac{1}{2c^2-8c+15}\le\frac{1}{3};$$

with equality for $a=3$ and $b=c=0$.

31. If a,b,c are the side-lengths of a triangle, then

$$a^3(b+c)+bc(b^2+c^2)\ge a(b^3+c^3).$$

32. Find the minimum value of $k > 0$ such that

$$\frac{a}{a^2+kbc}+\frac{b}{b^2+kca}+\frac{c}{c^2+kab}\geq\frac{9}{(1+k)(a+b+c)},$$

for any positive a, b, c. See the nice case $k = 8$.
Note. Actually, this inequality (with a, b, c replaced by $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$) is known.

33. If a, b, c, d are nonnegative real numbers such that

$$a+b+c+d=4, \quad a^2+b^2+c^2+d^2=7,$$

then

$$a^3+b^3+c^3+d^3\leq 16.$$

34. If $a \geq b \geq c \geq 0$, then

$$a+b+c-3\sqrt[3]{abc}\geq\frac{64(a-b)^2}{7(11a+24b)};$$

$$a+b+c-3\sqrt[3]{abc}\geq\frac{25(b-c)^2}{7(3b+11c)}.$$

35. If $a \geq b \geq 0$, then

$$a^{b-a}\leq 1+\frac{a-b}{\sqrt{a}}.$$

36. If $a, b \in (0, 1]$, then

$$a^{b-a}+b^{a-b}\leq 2.$$

37. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{24}{a^2b+b^2c+c^2a}+\frac{1}{abc}\geq 9.$$

38. Let x, y, z be positive real numbers belonging to the interval $[a, b]$. Find the best M (which does not depend on x, y, z) such that

$$x+y+z\leq 3M\sqrt[3]{xyz}.$$

39. Let a and b be nonnegative real numbers. Prove that

$$2a^2+b^2=2a+b\Rightarrow 1-ab\geq\frac{a-b}{3};$$

$$a^3+b^3=2\Rightarrow 3(a^4+b^4)+2a^4b^4\leq 8.$$

40. Let a, b and c are non-negative numbers. Prove that:

$$\frac{a+b+c+\sqrt{ab}+\sqrt{ac}+\sqrt{bc}+\sqrt[3]{abc}}{7}\geq\sqrt[7]{\frac{(a+b+c)(a+b)(a+c)(b+c)abc}{24}}.$$

41. Let a, b, c and d are non-negative numbers such that

$$abc + abd + acd + bcd = 4.$$

Prove that:

$$\frac{1}{a+b+c} + \frac{1}{a+b+d} + \frac{1}{a+c+d} + \frac{1}{b+c+d} - \frac{3}{a+b+c+d} \le \frac{7}{12}$$

42. Let a, b, c and d are positive numbers such that

$$ab + ac + ad + bc + bd + cd = 6.$$

Prove that:

$$\frac{1}{a+b+c+1} + \frac{1}{a+b+d+1} + \frac{1}{a+c+d+1} + \frac{1}{b+c+d+1} \le 1$$

43. Let $x \ge 0$. Prove without calculus:

$$(e^x - 1)\ln(1+x) \ge x^2.$$

44. Let a, b and c are positive numbers. Prove that:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{24\sqrt[3]{abc}}{a+b+c} \ge 11.$$

45. For all [b]reals[/b] a, b and c such that $\sum_{cyc}(a^2 + 5ab) \ge 0$ prove that:

$$(a+b+c)^6 \ge 36(a+b)(a+c)(b+c)abc.$$

The equality holds also when a, b and c are roots of the equation:

$$2x^3 - 6x^2 - 6x + 9 = 0.$$

46. Let a, b and c are non-negative numbers such that $ab + ac + bc \ne 0$.
Prove that:

$$\frac{(a+b)^2}{a^2+3ab+4b^2} + \frac{(b+c)^2}{b^2+3bc+4c^2} + \frac{(c+a)^2}{c^2+3ca+4a^2} \ge \frac{3}{2}.$$

47. a, b and c are [b]real[/b] numbers such that $a + b + c = 3$. Prove that:

$$\frac{1}{(a+b)^2+14} + \frac{1}{(b+c)^2+14} + \frac{1}{(c+a)^2+14} \le \frac{1}{6}.$$

48. Let a, b and c are [b]real[/b] numbers such that $a + b + c = 1$.
Prove that:

$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \le \frac{9}{10}.$$

49. Let a, b and c are positive numbers such that $4abc = a + b + c + 1$.

Prove that:

$$\frac{b^2+c^2}{a}+\frac{c^2+a^2}{b}+\frac{b^2+a^2}{c}\geq 2(a^2+b^2+c^2).$$

50. Let a, b and c are positive numbers. Prove that:

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{b}\right)\geq 1+2\sqrt[3]{6(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+10}.$$

51. Let a, b and c are positive numbers. Prove that:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{37(a^2+b^2+c^2)-19(ab+ac+bc)}{6(a+b+c)}.$$

52. Let a, b and c are positive numbers such that $abc=1$. Prove that

$$a^3+b^3+c^3+4\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)+48\geq 7(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

53. Let a, b and c are non-negative numbers such that $ab+ac+bc=3$. Prove that:

$$\frac{1}{1+a^2}+\frac{1}{1+b^2}+\frac{1}{1+c^2}\geq\frac{3}{2};$$

$$\frac{1}{2+3a^3}+\frac{1}{2+3b^3}+\frac{1}{2+3c^3}\geq\frac{3}{5};$$

$$\frac{1}{3+5a^4}+\frac{1}{3+5b^4}+\frac{1}{3+5c^4}\geq\frac{3}{8}.$$

54. Let a, b and c are non-negative numbers such that $ab+ac+bc\neq 0$. Prove that

$$\frac{a+b+c}{ab+ac+bc}\leq\frac{a}{b^2+bc+c^2}+\frac{b}{a^2+ac+c^2}+\frac{c}{a^2+ab+b^2}\leq\frac{a^3+b^3+c^3}{a^2b^2+a^2c^2+b^2c^2}.$$

55. Let a, b and c are non-negative numbers such that $ab+ac+bc=3$. Prove that

$$\frac{a+b+c}{3}\geq\sqrt[5]{\frac{a^2b+b^2c+c^2a}{3}};$$

$$\frac{a+b+c}{3}\geq\sqrt[11]{\frac{a^3b+b^3c+c^3a}{3}}.$$

56. Let a, b and c are non-negative numbers. Prove that

$$(a^2+b^2+c^2)^2\geq 4(a-b)(b-c)(c-a)(a+b+c).$$

57. Let a, b and c are non-negative numbers. Prove that:

$$(a+b+c)^8\geq 128(a^5b^3+a^5c^3+b^5a^3+b^5c^3+c^5a^3+c^5b^3).$$

58. Let a, b and c are positive numbers. Prove that

$$\frac{a^2-bc}{3a+b+c}+\frac{b^2-ac}{3b+a+c}+\frac{c^2-ab}{3c+a+b}\geq 0.$$

It seems that the inequality

$$\sum_{cyc}\frac{a^3-bcd}{7a+b+c+d}\geq 0$$

is also true for positive a, b, c and d.

59. Let a, b and c are non-negative numbers such that $ab+ac+bc=3$. Prove that:

$$a^2+b^2+c^2+3abc\geq 6.$$

$$a^4+b^4+c^4+15abc\geq 18.$$

60. Let a, b and c are positive numbers such that $abc=1$. Prove that

$$a^2b+b^2c+c^2a\geq\sqrt{3(a^2+b^2+c^2)}.$$

61. Let a, b and c are non-negative numbers such that $ab+ac+bc\neq 0$. Prove that:

$$\frac{1}{a^3+3abc+b^3}+\frac{1}{a^3+3abc+c^3}+\frac{1}{b^3+3abc+c^3}\geq\frac{81}{5(a+b+c)^3}.$$

62. Let m_a, m_b and m_c are medians of triangle with sides lengths a, b, c. Prove that

$$m_a+m_b+m_c\geq\frac{3}{2}\sqrt{2(ab+ac+bc)-a^2-b^2-c^2}.$$

63. Let a, b and c are positive numbers. Prove that:

$$\frac{a+b+c}{9\sqrt[3]{abc}}\geq\quad\frac{a^2}{4a^2+5bc}+\frac{b^2}{4b^2+5ca}+\frac{c^2}{4c^2+5ab}.$$

64. Let $\{a,b,c,d\}\subset[1,2]$. Prove that

$$16(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+a^2)\leq 25(ac+bd)^4.$$

65. Let a, b and c are positive numbers. Prove that

$$\sum_{cyc}\sqrt{a^2-ab+b^2}\leq\frac{10(a^2+b^2+c^2)-ab-ac-bc}{3(a+b+c)}.$$

66. Let a, b and c are non-negative numbers. Prove that:

$$\sum_{cyc}\sqrt{2(a^2+b^2)} \geq \sqrt[3]{9\sum_{cyc}(a+b)^3}.$$

67. Let a, b and c are positive numbers. Prove that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq \sqrt{\frac{15(a^2+b^2+c^2)}{ab+bc+ca}-6}.$$

68. Let a, b, c, d and e are non-negative numbers. Prove that

$$\left(\frac{(a+b)(b+c)(c+d)(d+e)(e+a)}{32}\right)^{128} \geq \left(\frac{a+b+c+d+e}{5}\right)^{125}(abcde)^{103}.$$

69. Let a, b and c are positive numbers. Prove that

$$a^2b+a^2c+b^2a+b^2c+c^2a+c^2b \geq 6\left(\frac{a^2+b^2+c^2}{ab+ac+bc}\right)^{\frac{4}{5}}abc.$$

70. Let a, b and c are non-negative numbers such that $a^3+b^3+c^3=3$. Prove that

$$(a+b^2c^2)(b+a^2c^2)(c+a^2b^2) \geq 8a^2b^2c^2.$$

71. Given real different numbers a, b and c. Prove that:

$$\frac{(a^2+b^2+c^2-ab-bc-ca)^3}{(a-b)(b-c)(c-a)}\left(\frac{1}{(a-b)^3}+\frac{1}{(b-c)^3}+\frac{1}{(c-a)^3}\right) \leq -\frac{405}{16}$$

When the equality occurs?

72. Let $x \neq 1$, $y \neq 1$ and $x \neq 1$ such that $xyz=1$. Prove that:

$$\frac{x^2}{\left(x-1\right)^2}+\frac{y^2}{\left(y-1\right)^2}+\frac{z^2}{\left(z-1\right)^2} \geq 1.$$

When the equality occurs?

73. Let a, b, and c are non-negative numbers such that $a+b+c=3$. Prove that:

$$a^5+b^5+c^5+6 \geq 3(a^3+b^3+c^3).$$

74.$a>1$, $b>1$ and $c>1$. Find the minimal value of the expression:

$$\frac{a^3}{a+b-2}+\frac{b^3}{b+c-2}+\frac{c^3}{c+a-2}.$$

75. For all non-negative a, b and c prove that:

$$(ab-c^2)(a+b-c)^3+(ac-b^2)(a+c-b)^3+(bc-a^2)(b+c-a)^3\geq 0.$$

76. Let a, b, c and d are positive numbers such that $a^4+b^4+c^4+d^4=4$. Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a}\geq 4.$$

Remark. This inequality is not true for the condition $a^5+b^5+c^5+d^5=4$.

77. Let a, b and c are positive numbers such that $\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}=3$. Prove that

$$\frac{1}{a+b}+\frac{1}{a+c}+\frac{1}{b+c}\leq\frac{3}{2}.$$

78. Let a, b and c are positive numbers such that $abc=1$. Prove that:

$$(a+b+c)^3\geq 63\left(\frac{1}{5a^3+2}+\frac{1}{5b^3+3}+\frac{1}{5c^3+2}\right).$$

79. Let a, b and c are positive numbers such that

$$\max\{ab,bc,ca\}\leq\frac{ab+ac+bc}{2},\quad a+b+c=3.$$

Prove that

$$a^2+b^2+c^2\geq a^2b^2+b^2c^2+c^2a^2.$$

80. Let a, b and c are positive numbers such that $a+b+c=3$. Prove that:

$$\frac{a^2}{3a+b^2}+\frac{b^2}{3b+c^2}+\frac{c^2}{3c+a^2}\geq\frac{3}{4}.$$

81. Let a, b and c are non-negative numbers and $k\geq 2$. Prove that

$$\sqrt{2a^2+5ab+2b^2}+\sqrt{2a^2+5ac+2c^2}+\sqrt{2b^2+5bc+2c^2}\leq 3(a+b+c);$$

$$\sum_{cyc}\sqrt{a^2+kab+b^2}\leq\sqrt{4(a^2+b^2+c^2)+(3k+2)(ab+ac+bc)}.$$

82. Let x, y and z are non-negative numbers such that $x^2+y^2+z^2=3$. Prove that:

$$\frac{x}{\sqrt{x^2+y+z}}+\frac{y}{\sqrt{x+y^2+z}}+\frac{z}{\sqrt{x+y+z^2}}\leq\sqrt{3}.$$

83. Let a, b and c are non-negative numbers such that $a+b+c=3$. Prove that

$$\frac{a+b}{ab+9}+\frac{a+c}{ac+9}+\frac{b+c}{bc+9}\geq\frac{3}{5}.$$

84. If x, y, z be positive reals, then

$$\frac{x}{\sqrt{x+y}}+\frac{y}{\sqrt{y+z}}+\frac{z}{\sqrt{z+x}} \geq \sqrt[4]{\frac{27(yz+zx+xy)}{4}}.$$

85. For positive numbers a, b, c, d, e, f and g prove that:

$$\frac{a+b+c+d}{a+b+c+d+f+g}+\frac{c+d+e+f}{c+d+e+f+b+g} > \frac{e+f+a+b}{e+f+a+b+d+g}.$$

86. Let a, b and c are non-negative numbers. Prove that:

$$a\sqrt{4a^2+5b^2}+b\sqrt{4b^2+5c^2}+c\sqrt{4c^2+5a^2} \geq (a+b+c)^2.$$

87. Let a, b and c are positive numbers. Prove that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{(4\sqrt{2}-3)(ab+ac+bc)}{a^2+b^2+c^2} \geq 4\sqrt{2}.$$

88. Let a, b and c are non-negative numbers such, that $a^4+b^4+c^4=3$. Prove that:

$$a^5b+b^5c+c^5a \leq 3.$$

89. Let a and b are positive numbers, $n \in \mathbb{N}$. Prove that:

$$(n+1)(a^{n+1}+b^{n+1}) \geq (a+b)(a^n+a^{n-1}b+\cdots+b^n).$$

90. Find the maximal α, for which the following inequality

$$a^3b+b^3c+c^3a+\alpha abc \leq 27$$

holds for all non-negative a, b and c such that $a+b+c=4$.

91. Let a, b and c are non-negative numbers. Prove that

$$3\sqrt[9]{\frac{a^9+b^9+c^9}{3}} \geq \sqrt[10]{\frac{a^{10}+b^{10}}{2}}+\sqrt[10]{\frac{a^{10}+c^{10}}{2}}+\sqrt[10]{\frac{b^{10}+c^{10}}{2}}.$$

92. Let a and b are positive numbers and $2-\sqrt{3} \leq k \leq 2+\sqrt{3}$. Prove that

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\frac{1}{\sqrt{a+kb}}+\frac{1}{\sqrt{b+ka}}\right) \leq \frac{4}{\sqrt{1+k}}.$$

93. Let a, b and c are nonnegative numbers, no two of which are zeros. Prove that:

$$\frac{a}{b^2+c^2}+\frac{b}{a^2+c^2}+\frac{c}{a^2+b^2} \geq \frac{3(a+b+c)}{a^2+b^2+c^2+ab+ac+bc}.$$

94. Let x, y and z are positive numbers such that $xy + xz + yz = 1$. Prove that

$$\frac{x^3}{1-4y^2xz} + \frac{y^3}{1-4z^2yx} + \frac{z^3}{1-4x^2yz} \geq \frac{(x+y+z)^3}{5}.$$

95. Let a, b and c are positive numbers such that $a^6 + b^6 + c^6 = 3$. Prove that:

$$(ab+ac+bc)\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) \geq 9.$$

96. Let a, b and c are positive numbers. Prove that

$$\sqrt[3]{\frac{a}{2b+25c}} + \sqrt[3]{\frac{b}{2c+25a}} + \sqrt[3]{\frac{c}{2a+25b}} \geq 1.$$

97. Let a, b and c are sides lengths of triangle. Prove that

$$\frac{(a+b)(a+c)(b+c)}{8} \geq \frac{(2a+b)(2b+c)(2c+a)}{27}.$$

98. Let a, b and c are non-negative numbers. Prove that

$$\sqrt[3]{\frac{(2a+b)(2b+c)(2c+a)}{27}} \geq \sqrt{\frac{ab+ac+bc}{3}}.$$

99. Let a, b and c are positive numbers. Prove that

$$\sqrt{\frac{a^3}{b^3+(c+a)^3}} + \sqrt{\frac{b^3}{c^3+(a+b)^3}} + \sqrt{\frac{c^3}{a^3+(b+c)^3}} \geq 1.$$

100. Let x, y and z are non-negative numbers such that $xy + xz + yz = 9$. Prove that

$$(1+x^2)(1+y^2)(1+z^2) \geq 64.$$

END.