103 學年度高級中學數學科能力競賽複賽試題

南區（台南區）　口 試 題

若兩正數α和β滿足$\log_9 \alpha = \log_{12} \beta = \log_{16}(\alpha+\beta)$，試求$\log_5 \dfrac{\alpha^2+\beta^2}{\alpha\beta}$之值。

【參考解答】:

令 $t = \log_9 \alpha = \log_{12} \beta = \log_{16}(\alpha+\beta)$

所以$\begin{cases} \alpha = 9^t \\ \beta = 12^t \\ \alpha+\beta = 16^t = 9^t + 12^t \end{cases}$

同除9^t得 $\left(\dfrac{16}{9}\right)^t = 1 + \left(\dfrac{12}{9}\right)^t = 1 + \dfrac{\beta}{\alpha}$

又$\left(\dfrac{16}{9}\right)^t = \left(\dfrac{4}{3}\right)^{2t} = \left(\dfrac{\beta}{\alpha}\right)^2$

令$x = \dfrac{\beta}{\alpha}$ 可得$x^2 = 1 + x$

因$x > 0$解之得 $\dfrac{\beta}{\alpha} = x = \dfrac{1+\sqrt{5}}{2}$ $\Rightarrow \dfrac{\alpha}{\beta} = \dfrac{2}{1+\sqrt{5}} = \dfrac{\sqrt{5}-1}{2}$

所以$\dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{\beta}{\alpha} + \dfrac{\alpha}{\beta} = \sqrt{5}$

故 $\log_5 \dfrac{\alpha^2+\beta^2}{\alpha\beta} = \dfrac{1}{2}$

口試 題

設a_1，a_2, …., a_n $(n \geq 3)$均為正實數，求$\sum_{i=1}^{n}\left[\frac{a_{i+1}+4a_{i-1}}{2a_i}\right]$的最小值為何?

(式子中令$a_0 = a_n$且$a_{n+1} = a_1$；符號$[x]$為不大於x的最大整數)。

【參考解答】$x-1<[x]$

所以$\sum_{i=1}^{n}\left[\frac{a_{i+1}+4a_{i-1}}{2a_i}\right] > \sum_{i=1}^{n}\frac{a_{i+1}+4a_{i-1}}{2a_i} - n \geq 2n-n=n$

故最小值為$n+1$

因為$\frac{a_2+4a_n}{2a_1}+\frac{a_3+4a_1}{2a_2}+\frac{a_4+4a_2}{2a_3}+\cdots+\frac{a_n+4a_{n-2}}{2a_{n-1}}+\frac{a_1+4a_{n-1}}{2a_n}$

$=(\frac{a_2}{2a_1}+\frac{2a_n}{a_1})+(\frac{a_3}{2a_2}+\frac{2a_1}{a_2})+(\frac{a_4}{2a_3}+\frac{2a_2}{a_3})+\cdots+(\frac{a_{n-2}}{2a_{n-1}}+\frac{2a_{n-3}}{a_{n-2}})+(\frac{a_1}{2a_n}+\frac{2a_{n-1}}{a_n})$

$\geq 2+2+\cdots+2=2n$