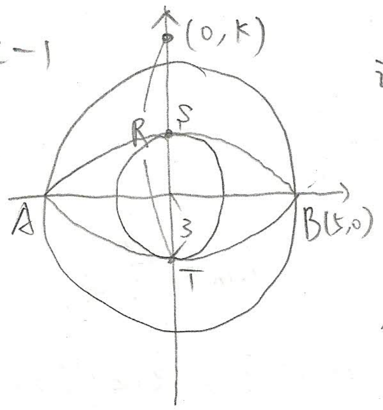
2012 TRML Individual by Fermat 2012.8.22.

I-1



設大圓eq: $x^2+y^2=25$ ，小圓eq: $x^2+y^2=9$

下半弧 $\widehat{ATB}$ 所在的圓eq: $x^2+(y-k)^2=R^2$

$\Rightarrow k=R-3$

$\Rightarrow x^2+(y-R+3)^2=R^2$ ，$B(5,0)$ 代入

得 $25+(R-3)^2=R^2 \Rightarrow 34=6R$ ，$R=\frac{17}{3}$ ※

I-2 令 $\log_x y=t \Rightarrow t+\frac{1}{t}=\frac{17}{4} \Rightarrow 4t^2-17t+4=0$

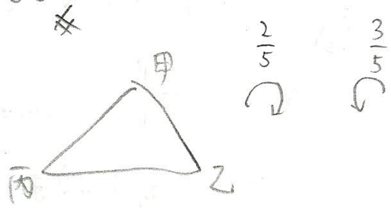
$\Rightarrow (4t-1)(t-4)=0 \Rightarrow t=\frac{1}{4} \vee 4 \Rightarrow y=x^{\frac{1}{4}} \vee y=x^4$

$(x=y^4)$

$\Rightarrow y^5=2^{10} \vee x^5=2^{10}$

$\Rightarrow \begin{cases} x=256 \\ y=4 \end{cases} \vee \begin{cases} x=4 \\ y=256 \end{cases} \quad \therefore x+y=260$ ※

I-3 設第i次的机率為 P_i ，轉移矩陣為 A



$\Rightarrow A=\begin{array}{c|c} & \begin{array}{ccc} 甲 & 乙 & 丙 \end{array} \\ \begin{array}{c} 甲 \\ 乙 \\ 丙 \end{array} & \begin{bmatrix} 0 & \frac{3}{5} & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ \frac{3}{5} & \frac{2}{5} & 0 \end{bmatrix} \end{array}$ ，$P_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow P_2=AP_1=\begin{bmatrix} 0 \\ \frac{2}{5} \\ \frac{3}{5} \end{bmatrix}$ ，$P_3=AP_2=\begin{bmatrix} \frac{12}{25} \\ \frac{9}{25} \\ \frac{4}{25} \end{bmatrix}$ ，$P_4=AP_3=\begin{bmatrix} \frac{35}{125} \\ \frac{36}{125} \\ \frac{54}{125} \end{bmatrix}$ $\frac{35}{125}=\frac{7}{25}$

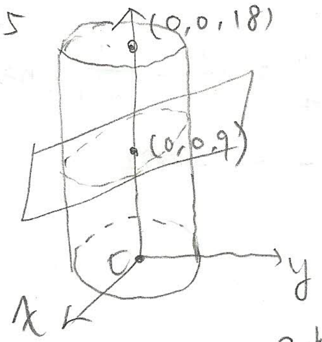
$\therefore \frac{7}{25}$ ※

I-4 $a>0$ ，$[(a+1)(a-2)]=1+5a \in \mathbb{N} \Rightarrow a=\frac{k}{5}$ ，$k \in \mathbb{N}$

$\Rightarrow 1+5a \le a^2-a-2 < 2+5a \Rightarrow \begin{cases} a^2-6a-3 \ge 0 \\ a^2-6a-4<0 \end{cases}$

$\Rightarrow \begin{cases} [a-(3+2\sqrt{3})][a-(3-2\sqrt{3})] \ge 0 \\ [a-(3+\sqrt{13})][a+(3-\sqrt{13})]<0 \\ a>0 \end{cases} \Rightarrow \underset{6.464\cdots}{3+2\sqrt{3}} \le a < \underset{6.6\cdots}{3+\sqrt{13}} \Rightarrow a=6.6=\frac{33}{5}$ ※

I-5



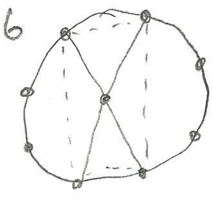
$2x+2y+z=9$ 過 $(0,0,9)$

考慮 $z=0$ 至 $z=18$ 間的圓柱体

平面將圓柱体積平分

$\therefore$ 所求体積 $=\frac{1}{2}(\pi \cdot 1^2) \cdot 18 = 9\pi$ ※

I-6



8點中任4點所成的弦中在圓內恰交一點

若三弦皆不共點，共有 $C^8_4=70$ 個交點，但三弦共點的交點有8個

故 $m=70-8 \cdot C^3_2+8=54$ ※

I-7 $2|a+1| + |b+2| = 3 \Rightarrow \begin{cases} |a+1| = 0 \\ |b+2| = 3 \end{cases}$ 或 $\begin{cases} |a+1| = 1 \\ |b+2| = 1 \end{cases}$

∴ (a,b) 共有 $1\times 2 + 2\times 2 = 6$ 組解 ※

I-8 $5^{x^2+4y} + 5^{4y^2+2z} + 5^{z^2+2x} \geq 3\sqrt[3]{5^{x^2+4y^2+z^2+2x+4y+2z}}$

$= 3\sqrt[3]{5^{(x+1)^2+(2y+1)^2+(z+1)^2-3}} \geq 3\sqrt[3]{5^{-3}} = \frac{3}{5}$ ∴ $\min = \frac{3}{5}$ ※

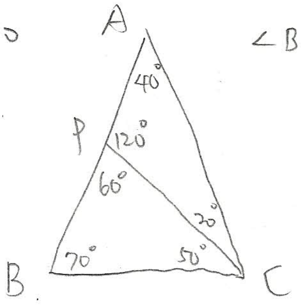
I-9 2, 2, 2, 3, 3, 5, 5 取至少2球的乘積，求32種相異乘積值總和

考慮 $n = 2^3 \cdot 3^2 \cdot 5^2$ 的正因數，共有 $(3+1)(2+1)(2+1) = 36$ 個

扣除 1, 2, 3, 5 無法以兩球乘積表示，故此32種相異值的和為

$(1+2+2^2+2^3)(1+3+3^2)(1+5+5^2) - (1+2+3+5) = 15\cdot 13\cdot 31 - 11 = 6034$ ※

I-10



$\angle B = \angle ACB = 70°$, $\angle ACP = 20°$, $\angle BPC = 60°$

$\frac{\overline{AP}}{\overline{CP}} = \frac{\sin 20°}{\sin 40°}$, $\frac{\overline{CP}}{\overline{BC}} = \frac{\sin 70°}{\sin 60°}$

兩式相乘得 $\frac{\overline{AP}}{\overline{BC}} = \frac{\sin 20° \cos 20°}{\frac{\sqrt{3}}{2}\sin 40°} = \frac{\frac{1}{2}\sin 40°}{\frac{\sqrt{3}}{2}\sin 40°} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ ※

I-11 $f(x) = 2x^2 + \alpha x + \beta$ 過 $(r,0)$, $(s,0)$, $(5,-358)$, $r, s \in \mathbb{N}$, $r \neq s$

不妨令 $r < s$, $\begin{cases} r+s = -\frac{\alpha}{2} \\ rs = \frac{\beta}{2} \end{cases}$

$f(5) = -358 \Rightarrow 5\alpha + \beta = -358 - 50 \Rightarrow 2rs - 10(r+s) = -408$

$\Rightarrow rs - 5(r+s) + 25 = -204 + 25 \Rightarrow \underbrace{(r-5)}_{\geq -4}\underbrace{(s-5)}_{\geq -4} = -179 = (-1)\times 179$

$\Rightarrow r = 4$, $s = 184$

$\Rightarrow \beta = 2rs = 2\cdot 4\cdot 184 = 1472$ ※

I-12 $a_1 = \frac{1}{7}$, $\frac{a_{n-1}}{a_n} = \frac{4n a_{n-1} + 1}{1 - 4a_n}$, $n \geq 2$, $a_{30} = ?$

$4n\, a_n a_{n-1} + a_n = a_{n-1} - 4a_n a_{n-1}$

$\Rightarrow 4(n+1)\, a_n a_{n-1} = a_{n-1} - a_n \Rightarrow 4(n+1) = \frac{a_{n-1} - a_n}{a_n a_{n-1}} = \frac{1}{a_n} - \frac{1}{a_{n-1}}$, $n \geq 2$

$\Rightarrow \sum_{n=2}^{30}\left(\frac{1}{a_n} - \frac{1}{a_{n-1}}\right) = \sum_{n=2}^{30} 4(n+1) = 4\sum_{k=3}^{31} k = 4\left(\frac{31\cdot 32}{2} - 1 - 2\right) = 1972$

$\Rightarrow \frac{1}{a_{30}} - \frac{1}{a_1} = 1972$, $\frac{1}{a_{30}} = 1972 + 7$

$\Rightarrow a_{30} = \frac{1}{1979}$ ※